RESONANCE IN THE SOLAR SYSTEM*

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While the title of this address implies limited applications of resonance phenomena, it is necessary, in order to develop those applications, to consider the foundations on which the theory of resonance rests. The foundations are partly physical and partly mathematical since we have to consider not only the phenomena but the symbolic representation of them. At the very beginning of the discussion it is seen that questions concerning the degree of accuracy of physical measurements are involved in a fundamental manner. A wide range of topics has thus to be brought into the argument and some considerable time must be spent on quite elementary details if the ideas to be developed are to be made clear. The physicist will recognize applications in many directions, but I shall confine the latter mainly to those which arise in the motions of bodies within the solar system.

In order to make clear the questions with which we have to deal, it is necessary to understand what we mean by *periodicity*. Take the case of simple harmonic motion defined by

$$x = a \sin (bt - c).$$

In all physical problems, a, b, c are measured quantities deduced directly or indirectly from observation. If small errors occur in a, c only, the change in x is always small, however great t may be. But a small error in b induces finite changes in x when t is sufficiently great. As our power of measuring b is always limited, the time during which we can predict the value of x is also limited. Further, the conditions under which mechanical systems operate are such

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that b itself will not remain permanently the same, unless it be a universal constant of nature, and we know of no such constant as far as the motions of finite masses are concerned. The same result must follow for any periodic motion. The mathematical definition of periodicity is thus not sufficient. In the applications we must regard it as an average or temporary phenomenon. This point of view has to be emphasized when, in celestial mechanics, we use the phrase "mean angular velocity" which implicitly assumes that such a mean exists and is determinate. Where we are dealing with phenomena over a limited time interval, the matter is not usually so important, but when our interval becomes very great as in questions of cosmogony it must be taken into account.

The word *resonance* is usually used to describe the phenomena which are observed when, in some mechanical system in motion, the periods of oscillation of two of its parts, not rigidly connected with one another, become the same. It would seem at first sight that the definition has no meaning, for the slightest change in our measures of the periods would abolish the resonance. This difficulty partly disappears when we learn that, in general, it is not necessary to have the periods exactly equal in order to produce the phenomena; it is only necessary that their difference shall be less than some finite amount. In the language of mathematics, the phenomena of resonance belong, not to a point, but to a finite range.

When we remember the fundamental importance of resonance in all physical observations—indeed our capacity to see and hear and therefore to observe depends on its existence—it is somewhat strange how little concerning the actual phenomena is to be found in the textbooks and treatises which deal with physical problems. Partly this is due to the fact that damping due to friction occurs in our mechanical systems and damping prevents the appearance of the essential characteristics of the motion. Partly also it is due to the general use of linear systems of differential equations which give results differing fundamentally from those of non-linear systems. In the resonance problems of celestial mechanics we have no damping, the equations are not linear in general even in a first approximation, and every "constant" present is a measured constant and subject to both error and change. Since a considerable degree of delicate adjustment is required to produce resonance, it is evident that these factors must be taken into consideration when discussing the mathematical representation of the phenomena.

Let us take the linear system

$$\ddot{x} + n^2 x = a \sin(pt + \alpha).$$

This represents a motion which is the sum of two harmonic oscillations of periods $2\pi/n$, $2\pi/p$, as long as $n \neq p$. Since the amplitude of the second oscillation is $a \div (n^2 - p^2)$, this amplitude increases indefinitely as $n-p$ approaches zero. When $n-p=0$ exactly, the solution changes its character and can be represented by two harmonic terms one of which has an amplitude of the form bt where b is finite. But from the physical point of view there is no discontinuity in passing from $n-p$ very small to $n-p=0$: the oscillations increase in amplitude as $n-p$ approaches zero.

For a linear system with several variables

$$\ddot{x}_i + \sum_j a_{ij} x_j = 0, \qquad (i.j = 1, 2, \cdots, n),$$

in which the constants a_{ij} are such that each coordinate can be represented as the sum of n harmonic terms with n different periods, it can be shown that a similar increase in the amplitudes occurs, when two of the periods become nearly equal. An exception to this result arises when we can, by a linear change of the variables, separate the system into two independant sets in one of which $i, j=1, \cdots, p$ and in the other, $i, j=p+1, \cdots, n$.

These results, however, are of little value when the systems are not linear. Both the physical characteristics of the ensuing motions and their mathematical representation are changed. We cannot lay down any general type of equation, as in the linear case, and apply it to the particular cases which may be presented. The better plan therefore seems to be to take a special type and discuss it in some detail. But before doing so, I must try and give some indications of the difficulties which planetary and satellite motions in the solar system present, and of the methods which at present are in use to attack them.

The problem of the motion of three or more bodies under the Newtonian laws of motion and gravitation is so complicated that I cannot hope even to give an outline which shall be intelligible in the course of an hour's talk. And it would take me far longer to lead to the particular phenomena which form the subject matter of this lecture by any logical process even if the mere steps were simply outlined and explained. I shall therefore content myself with presenting the problems involved in the form of mathematical or physical analogies which will have the additional advantage of corresponding to simple physical mechanisms with which everyone is familiar.

I have two main ideas in view in discussing the question of resonance. One, the more immediate, is an attempt to deal with a set of observed phenomena which have not as yet received a fully adequate explanation, or perhaps, to be a little more specific, to account for the phenomena on the law of gravitation only without invoking special hypotheses as to the original formation of the solar system. This first question is the apparent avoidance of resonance in certain cases of planetary motions and the apparent permanence of resonance in others. The course of the discussion is seen to lead inevitably to the two parts of the second question—the configuration of the system in the distant past and distant future. Can we learn anything about the first part, usually termed the origin of the solar system, from its present configuration? And to what extent can we predict the future? Has the solar system the property which is usually but rather loosely, termed "stability"?

"Stability" in the usual sense is a mathematical rather than a physical property of an actual mechanical system. It is not difficult to deal with when the system is at rest, or when we confine its motion to certain very simple configurations. When we come, however, to deal with the problem of three or more bodies, it can have almost any meaning we like to attach to it. One common meaning in celestial mechanics is the limitation of one of the elements, say the velocity or mean distance, to certain limits. If we had any exact knowledge of what those limits were, there would be some satisfaction in using the term.

Let me give an illustration. One of the theorems often quoted is that which refers to the "stability" of the eccentricities. It is said that it can be shown that the eccentricities of the planets always lie between certain limits. What does this assume? The variables which define the motion are changed from the ordinary coordinates to new variables which are those defining the position, shape, and size of what is called the osculating ellipse. If there were only two bodies involved, this ellipse would have fixed position, shape, and size. When other bodies attract, these elements vary. Suppose we confine the variations to the first powers of the changes from constancy, as is done in the mathematical discussions on stability. Then the total change is the sum of all the individual changes. The theorem in question simply refers to the individual changes as calculated in this manner; it is the usual "mathematical stability." Each of them is periodic and therefore limited in magnitude. The proof, however, breaks down if we proceed to higher powers of the changes when there is approximate or exact resonance between two or more of the periods present. The limits between which the eccentricities can vary may become so wide as completely to alter the general configuration of the system; from the point of view of the physicist the motion is ultimately unstable.

In the ordinary discussions on stability we ask what is the result of slightly changing the position and velocity at any instant. In resonance problems we have to ask what happens when we introduce small periodic disturbing forces; at least it is in this way that the problems of celestial mechanics often present themselves. The physicist accustomed to the phenomenon in most of his work is able to say at once that the oscillations of the system increase in amplitude, and this indeed may be regarded as the first general effect of resonance. What happens later depends on the construction of the system. In terrestrial mechanisms we are accustomed to observe the early increase of amplitude and the apparent change of period which accompanies it. Frictional or damping forces are, however, always present, and since these affect resonance phenomena in a fundamental manner it is not easy to use familiar illustrations in order to visualize what will happen in the absence of friction. In atomic systems, provided the periodic disturbing force is not too small, the description usually is confined to a statement that the type of motion changes. Atomic systems under their modern descriptions have certain resemblances to the problems of celestial mechanics. But there is this difference. In the atomic problems we observe the aggregate or average effect of millions of revolutions of the electrons about the nucleus but have no method of observing the positions in the orbits at any one time. In the solar system, we can observe the latter, but so far have not been able to calculate the result of millions of revolutions.

Let us now proceed to the physical analogy which will serve to illustrate some of the phenomena of resonance in the solar system. It depends on the fact that the mathematical expressions are somewhat similar when reduced to their simplest forms. As with most analogies, however, the correspondence is not complete and must not be pushed too far. The analog to be used is that of the motion of the common pendulum. But the pendulum I have in mind differs from that attached to a clock in that it can make complete revolutions. The mechanism is perhaps better thought of as a bicycle wheel mounted on a stand with ball bearings so as to be as nearly frictionless as the imagination can make it; the actual description supposes that there is an entire absence of friction. To one point of the rim an iron weight is clamped. The wheel can be set in motion in various ways and can be acted on by different types of forces; some of them can be supposed to be impressed by a moving magnetic field which can act on the iron weight.

When no external force acts the motion of such a pendulum is well known. If it be given a small tap it oscillates in a period which depends on the amplitude; there is no such thing physically as an infinitesimal tap which gives a period independent of the amplitude. The greater the amplitude, the longer the period. The limit of the latter is infinite when the pendulum just reaches the highest point. The motion with a larger tap changes its character; the pendulum executes complete revolutions in either sense, with a variation of velocity between the highest and lowest points.

Mathematically, the motion is expressed by the equation

$$\ddot{x} + k^2 \sin x = 0,$$

where x is the angle between the position of the pendulum at any time and its equilibrium position at the lowest point. As long as the pendulum is oscillating, its motion is expressed by

$$x = \text{periodic function of } t, \text{ period } 2\pi/n.$$

When it is making complete revolutions, the motion is given by

$$x = \pm nt + A + \text{periodic function of } t, \text{ period } 2\pi/n.$$

In each case the period is an arbitrary constant of the solution depending on the initial impulse. It approximates to $2\pi/k$ only when the impulse is very small. Separating the two solutions is one in which x is a discontinuous (indeterminate) function of t at the highest point.

When the motion is near this critical case, a very small change in the initial conditions will not only completely alter the physical character of the motion but it will do so in a finite time. That is to say, the nearer the pendulum to the critical case, the shorter is the interval during which we can predict its position with a given degree of accuracy. The comparison here is with the non-critical case where the accuracy of prediction is more nearly proportional to the accuracy with which we can measure the constants.

Suppose that while the pendulum is executing small oscillations a periodic force with a period nearly equal to $2\pi/k$ acts. We know what happens. The oscillations begin to increase and, after the amplitude reaches a maximum, begin to decrease again, the process being repeated as long as we are able to observe it. A force of this type might be expressed by

$$\ddot{x} + k^2 \sin x = a \sin (kt + \alpha).$$

The phenomena of resonance in this case may be described by saying that the amplitude increases until the average period is sufficiently different from $2\pi/k$ as to change the relative phases by 180°, when the amplitude begins to decrease again owing to the external force opposing the motion.

But if we ask whether this process is repeated indefinitely, the answer is not so simple. We may regard the system as having two periods, $2\pi/k$ and another period which approximately measures the time between maximum amplitudes. The motion cannot be exactly recurrent unless these two periods are in the ratio of two whole numbers. When this happens the equation must have a periodic solution. This would evidently be a special case and we are desirous of finding out what happens under *any* initial conditions. A way out of the difficulty is to suppose that we can always find a periodic solution.

Suppose that the two frequencies mentioned above were k, k'. The last hypothesis demands that we shall always be able to find two integers, i, j, such that $k/k' = i/j$. Suppose that we have found such a case and that we either slightly alter the initial conditions or, what amounts to the same thing, our measures from observation of k or k'. The ratio k/k' would then be i'/j', where i', j' are two new integers.

Now while the difference $i/j-i'/j'$ may be very small, the theory of convergents in continued fractions teaches that $i-i'$ and $j-j'$ will be widely different, and, in general the smaller the difference between the two fractions, the greater will be the differences between their numerators and their denominators. Since the whole periods of the motions are $2\pi i/k$, $2\pi i'/k$ in the two cases, the period is discontinuous; there is a finite change in it corresponding to an infinitesimal change in the initial conditions.

It may still happen, however, that the amplitude of x always remains less than some given quantity. Suppose we attempt to solve the equation by means of infinite series of harmonic terms. The arguments will be of the form $(ik \pm jk')t+$const., where i, j are integers. The process of integration gives coefficients of the form $a_{ij} \div (ik \pm jk')$, where a_{ij} diminishes to zero as i or j approach infinity. In general, the values of a_{ij} change only by an infinitesimal amount when k or k' similarly changes, so that it would seem always possible so to choose the measures of k, k' that the series is finite or that it shall be infinite. This means that the motion is either physically indeterminate within a finite time, or that our assumption of a solution oscillating between finite limits must be abandoned.

The problem is similar to that of the integral

$$\int \sum a_{ij} \sin (ikt \pm jk't + \alpha_{ij}) dt, \qquad (i,j = 1,2, \cdots ,\infty),$$

where $\sum |a_{ij}|$ is convergent. The result is infinite or indeterminate as long as k, k' are measured quantities. The only possible conclusion appears to be that, except for special cases, at some time in its motion the pendulum will begin to make complete revolutions. Incidentally, it is interesting to note that the small amount of friction present in the motion of the most carefully constructed pendulum is necessary in order that it may furnish an accurate measure of time.

Let us next consider what happens when complete revolutions are under way. The difficulties caused by the assumption of a doubly infinite series still appear to be present, that is to say, we cannot assume that the motion is expressed by $x = k_1 t +$ an oscillating function of t. In general, however, the average speed of the complete revolutions will not continuously increase but will reach a maximum and then diminish again. When they have reached the point where oscillations can take place, there are two possibilities, namely, oscillations, or complete revolutions in the opposite sense. But since a limitation of the phase of the motion is necessary in order that oscillations may be set up, the reversal of the complete revolutions will be the usual phenomenon and only after many reversals will oscillation be again set up. These, after continuing for a time, will again be changed to complete revolutions, the maximum speed of which is apparently not limited, but will itself have oscillations the extent of which will be greater at certain intervals, the longer the motion continues.

The times occupied by these various phases of the motion will depend on the values of the constants present in the motion. We can get some guide to these from the general principle that the body will remain longest in that phase of motion which has the greatest degree of relative stability, that is, where small changes in the constants will produce the least effect in a given interval of time. These appear in general to be either the configurations where the oscillations are at their minimum amplitude or where the revolutions have their maxima of average speed, or at both. In certain cases of celestial mechanics in which the mechanism is too complicated for the use of the pendulum as an approximate analogue, small oscillations are impossible, and for these the phases corresponding to the maximum speeds of revolution will be the configuration in which most of the time will be spent.

The general description of the motion of a pendulum under a periodic force $a \sin (pt+\alpha)$ is not essentially different from that under a force

$$a \sin (x + pt + \alpha),$$

which resembles more closely some of the cases in celestial mechanics, especially when p/k is small. This latter is the case of a periodic force which has nearly the same period as the pendulum for all amplitudes except for those which approach complete revolutions. It can be shown that unless a/pk is small, the latter state is in general set up in an interval comparable with $2\pi/p$. A relatively stable configuration is not reached until the average angular velocity of revolution considerably exceeds p. When a/pk is small, the time required to produce complete revolutions may be very long. The distinction between the cases a/pk large and small appears to have an important bearing on the asteroid problem.

It has been stated that, in general, the motions which have the greatest relative stability are those in which the average speed of revolution considerably exceeds p. The ordinary processes of approximation show that in this motion, the periodic change in the average angular velocity as the pendulum goes round varies as k^2. As the speed diminishes, this amplitude tends to vary as the first power of k. In the applications to celestial mechanics, the unit of time is the period of revolution of the body,* and in all such cases k^2 is a quantity small compared with unity. Thus k^2 is small compared with k. Since resonance corresponds to the case where the pendulum is oscillating, it is convenient to define a "range" attached to the resonance and to measure it by k. Thus resonance phenomena take place within the range k; outside this range ordinary oscillations or perturbations, as they are usually called, occur.

I have dealt hitherto with the cases where the motion is analogous to a pendulum which can make very small

* This must not be confused with the period of revolution of the pendulum.

oscillations when no disturbing force acts, and have mentioned the possibility of cases of motion in which this analogy breaks down to the extent that small oscillations are impossible, while oscillations of finite amplitude do appear to be possible. It is evident that in this event the change from oscillations to complete revolutions under disturbing forces of the form $a \sin(x+pt+\alpha)$ will take place more rapidly and that the time during which oscillations are under way will be relatively shorter. This result may be put in a form more useful for the applications by the statement that the time spent in oscillation compared with that spent in complete revolutions is very much shorter, and consequently over any long interval the chance of observing oscillations is smaller.

In the whole of this discussion it is assumed that there is a complete absence of frictional forces. The moment we introduce them, however small in amount, the physical results are changed fundamentally. It would seem that the most familiar resonance effect—the identity of the periods of rotation about its axis and revolution around the earth of the moon—could not persist without large oscillations (librations) in the absence of small viscous forces in the moon's mass. The first effect of a non-uniform change in the period of revolution is to increase the amplitude of these free oscillations or librations. But the lunar bodily viscous forces tend to damp them down if the change is very slow. It follows as a corollary that once this resonance has been set up, it may continue indefinitely. Bodily viscosity in the moon's mass therefore changes a configuration which is unstable in the presence of small periodic disturbing forces into one in which the stability is permanent. Thus the moon will always turn the same face to the earth. A similar effect keeps the Eulerian nutations of the earth's axis small.*

My main object, however, is the consideration of resonances amongst planetary and satellite motions where

* H. Jeffreys, *The Earth*, p. 248.

we assume a complete absence of frictional forces, each body being treated as a particle with complete rigidity. I shall first consider an asteroid of very small mass moving in a planetary orbit round the sun, and acted on by planets of large masses, Jupiter and Saturn for example, the latter however, having masses small compared with that of the sun.

If the asteroid were not acted on by other planets it would describe an ellipse about the sun with constant period $2\pi/n$ and the angle described after any time would be given by

$$v = w + \text{periodic function of } w, \quad w = nt + \text{const.},$$

the distance being a periodic function of w having a finite mean value. The "elements" of the ellipse are usually taken to be w, the mean distance, the eccentricity, and three further elements which define the position of the ellipse, and the position of the asteroid in the ellipse at some given moment. When Jupiter acts on the asteroid, one useful method is to find how these elements vary. Suppose Jupiter is moving in a fixed ellipse, the mass of the asteroid being too small to affect the motion of Jupiter sensibly. Then the motion of Jupiter is expressible as a function of $w' = n't + \epsilon'$, the other elements being constant.

The only way we have to find the motion of the asteroid under these circumstances is to proceed by some method of continued approximation. When we do this, the first approximation to w is given by an equation of the form

$$\ddot{w} = mn^2 \sum i a_{ij} \sin(iw + jw' + \beta_{ij}),$$

where $i, j = 0, \pm 1, \pm 2, \cdots$, giving an infinite series. The magnitudes of the coefficients mainly depend on a parameter e which is of the order of the eccentricities and inclination. In general, a_{ij} is divisible by $e^{|i+j|}$. The factor m is the ratio of the mass of Jupiter to that of the sun and is about .001.

Having decided what accuracy we need, we cut off the series at some definite values of i, j. We then put $w = nt + \epsilon$,

$w' = n't + \epsilon'$ in the right hand member and integrate, obtaining

$$w = nt + \epsilon - mn^2 \sum \frac{ia_{ij}}{(in + jn')^2} \sin (iw + jw' + \beta_{ij}).$$

This method gives a good approximation so long as $in + jn'$ is not too small.

Suppose, however, that $in + jn'$ in one term is so small that the corresponding coefficient is very large and that even the oscillation of the mean angular velocity, w, is large. Then evidently the approximation is useless.

To avoid the difficulty put

$$iw + jw' + \beta_{ij} = x + \pi \text{ or } x$$

according as a_{ij} is positive or negative, and omit all the other terms as being small compared with this. The equation for x then takes the form

$$\ddot{x} + k^2 \sin x = 0.$$

Since the integral of this equation is

$$\dot{x}^2 = b + 2k^2 \cos x,$$

the value of the angular velocity is limited whatever be the initial conditions. So far the presence of this resonance term produces no infinity. But the character of the motion is changed similarly to that of the pendulum when it changes from making oscillations to complete revolutions.

Suppose now that Saturn is present. It is large enough to affect the motion of Jupiter sensibly and introduces terms of the form $a \sin(x + pt + p')$ where $2\pi/p$ is a period which may be very long compared with the period of revolution of the asteroid. If this is simply an addition to w' its effect is not usually difficult to deal with for reasons into which I shall not enter. But there are certain terms which produce large periodic changes in the eccentricity of Jupiter—the so-called secular terms— and these terms appear to compel us to deal with equations of the form

$$\ddot{x} + k^2 \sin x = a \sin (x + pt + \alpha).$$

As long as a/pk is small, we can continue our approximations, and it may be that the higher approximations will avoid the phenomena which have been explained earlier, although this is unlikely. In any case, however, the oscillation, if started with x small, may continue for long periods of time before x can reach the values $\pm\pi$. But if a/pk is large, the oscillations must soon turn into complete revolutions and continue until the asteroid is well outside the resonance range. Once there, as explained earlier, it will at least remain for a long time even if it can get back at all.

From the point of view taken here, the most important group of asteroids is that known by the name "Trojan," from the circumstance that all its six known members have received names famous in the Iliad of Homer. The mean period of each of these round the sun is at present the same as that of Jupiter, that is, the resonance is due to the equality of n, n', and consequently the resonance terms are those for which $i=-j$. The attendant circumstances, namely, that they are at the same mean distance from the sun and that they oscillate about the third vertices of the two equilateral triangles which have Jupiter and the sun as the other two vertices, are incidental to this discussion, except in so far as they render the discussion possible. Laplace, a century ago, showed that an ideal exact solution of the problem of three bodies was a configuration in which all of them remained at the corners of an equilateral triangle while describing ellipses of the same size and shape. It has further been shown that under infinitesimal displacements from this ideal configuration these deviations will remain infinitesimal, under conditions as to the magnitudes of the masses which are here satisfied.

Now the set of deviations in which we are interested corresponds to the case of the pendulum performing infinitesimal oscillations. But the observed deviations are far from being of this character; in the case of one asteroid they correspond to an amplitude of the order of 20° from the vertical position.

If we continue the approximations on the supposition that Jupiter moves in a fixed elliptic orbit, there is nothing to indicate, within the extent to which these approximations can be carried with a reasonable amount of labor, that the configuration is otherwise than stable for very long intervals. When, however, we introduce the action of Saturn, both directly on the asteroid and indirectly through its effect on Jupiter, we introduce large perturbations, chiefly because these two planets, and therefore Saturn and an asteroid, have their periods nearly in the ratio 2:5. It can be shown that the principal part of this action—that arising in the mean angles w, w'—is transferred directly to the asteroid. But there are other portions constituting slow changes in the eccentricity of Jupiter which introduce terms corresponding to the disturbing force $a \sin(x+pt+\alpha)$ in the motion of the pendulum. The number a/kp for these terms is small as long as the eccentricity of the asteroid is not too great. In the case of one asteroid, Achilles, I was able nevertheless to show* that a very slight change in the conditions would produce large oscillations (a secondary resonance) in its eccentricity—so large indeed that they might bring the asteroid to a close approach to Jupiter. Changes of this character are always furnished by the attractions of other planets, granted sufficient time, so that the present apparent stability of the orbit may be lost in the course of time.

The next great group is that of the 2:1 ratio. The first approximation for this group with Jupiter alone acting is such that small oscillations are impossible but oscillations with finite amplitude probably can exist. When we include the action of Saturn, the analogue of the case in which a/kp is large comes into play. We therefore expect that the time spent in oscillations is small compared with that spent in making complete revolutions. The "range" of the resonance is approximately $(0.0007e)^{1/2}$ where e is the average eccentricity.† If e be 0.1, this gives us a range of period for

* Transactions, Yale Observatory, vol. 3, p. 114.

† Monthly Notices, Royal Astronomical Society, vol. 72, p. 619.

resonance conditions of the order of fifty days about the half period of Jupiter. Within this range very few asteroids should be found at any one time. There is another feature about this resonance point which can be deduced from the equation, namely, that oscillations can only be set up from complete revolutions in one sense; this of course statistically cuts down the probable number by one half.

In all resonance cases except that of the 1:1 ratio the range of the eccentricities is large. In the case of the 2:1 group, in whatever manner the asteroid be started, the eccentricity at some time must be at least 0.2.* Near the edge of the range is a resonance point 5:1 with Saturn, and though the range of this new resonance point is much smaller, it will add to the improbability of finding an asteroid within the 2:1 range at any time. These results agree well with the statistical data. A considerable number of asteroids are known which can be well interpreted as having been nearly uniformly distributed near this resonance point, but there is a complete (with one or two doubtful exceptions) absence of asteroids within the range. Further, there appear to be more than the average number just outside the range.

The equations for all the resonance points $(1+i):i$ are quite similar to those of 2:1 with a slow decrease in range as i increases. There are, however, few asteroids near these ranges, so that statistical results are not properly applicable. But with higher values of i, two facts must be noted. In the first place, the resonance points become more crowded together as i increases and they approach the 1:1 resonance as a limit. In the second place, the mean distances approach that of Jupiter with increase of i and a circumstance of which little account has been yet taken, namely, a very close approach to Jupiter becomes more probable. Such a close approach fundamentally changes the orbit and more particularly the eccentricity, so that asteroids which have experienced this phenomenon must be regarded as belonging

* Loc. cit., Monthly Notices, p. 629.

to a somewhat different statistical group. The above considerations, however, indicate that the chance of an asteroid remaining near these higher resonance points diminishes rapidly with the increase of i, or in other words, the action of Jupiter and Saturn will clear them away much earlier than those near the 2:1 ratio. Finally, while the resonance points $i:j$ for small values of i, j are those most likely to be effective in showing resonance phenomena, the values of n'/n just outside these resonance ranges (except the 1:1 case) are the most stable since, when the ranges are taken into account, these positions are the least crowded of all by resonance phenomena.

Besides the asteroids, there is a family of comets which have also periods nearly half that of Jupiter. On account of their large eccentricities the methods used for the asteroids are not available. But it would seem that the same general results will hold. During the writing of this paper, the conclusions reached were illustrated in a publication by Professor A. O. Leuschner* on the Pons-Winnecke Comet. This comet in 1819 had a mean daily angular velocity round the sun of 632″ which increased to 639″ in 1858 and then diminished to 604″ in 1915. Between that date and 1927 it crossed the 2:1 resonance point of 599″ and is now 591″. This extraordinarily rapid change is partly "incidental" and due, as Leuschner points out, to the fact that the aphelion distance and phase are such that, during the last century, the comet has approached Jupiter very closely every other revolution. Theory suggests that these close approaches cannot persist through very many more revolutions. The point, however, namely that a large perturbation can carry a body rapidly through the resonance region, seems to be exemplified in this case.

Other resonance points within the group of asteroids are treated in the same general manner. In the case of the 3:1 group, small oscillations appear to be possible. The range of

* Publications of the Astronomical Society of the Pacific, October, 1927.

the resonance is smaller and the statistical material less plentiful with a smaller region of resonance; however, the disturbing effects have more power; in fact, in our pendulum analogy the terms of the right-hand member are of the same order of magnitude as those on the left, and, as stated above, the time which an asteroid can spend in this region is correspondingly less, giving a smaller chance of finding an asteroid there at any one time.

The rings of Saturn furnish valuable statistical material. Here the millions of "stones" or "rocks" which must constitute those rings revolve round Saturn and resonances are caused by the action of its larger satellites. There is little doubt that the well known divisions of the rings are caused by resonance action. But unless my main thesis is correct, namely that the time spent in oscillation within the resonance region is very short compared with the time spent outside, it is difficult to account for these divisions. Unfortunately we cannot at present observe the motion of individual members of the rings. We can, however, obtain a measure of the number at a given distance from Saturn by photometric observations and this number should be indicated also by the theory. The mathematical treatment, while very difficult and full of pitfalls, does not appear to be outside the range of present possibilities. It must, however, be carried out on practical lines, that is, with full consideration of the observational data.

I have up to now dealt with cases in which the integrals of the problem of three or more bodies are of little or no assistance on account of the small masses of the disturbed bodies in comparison with those of the bodies which produce the perturbations. Outside the integrals which define the motion of the center of mass of the system and which play no part in this discussion, there are only four known, or, if we confine the motion to one plane, only two, namely, the integrals of energy and angular momentum.

Since the portion contributed by a body to any one of these integrals has the mass of that body as a factor, it is at once evident that very small changes in the momentum or energy of a large body can change fundamentally those of the small bodies, as far as these integrals are concerned. From the point of view taken here, an exact integral constitutes a limitation on the range of variation. This may also be interpreted by the statement that the existence or near existence of an integral lengthens the time during which a given configuration can persist. When the masses present in the integral are of the same order of magnitude we can make use of it. Thus the mutual action of Jupiter and Saturn can be treated with the use of these integrals because the actions of the other planets in the solar system affect the integrals only to a comparatively small degree in a limited time. Similarly, in testing the mutual actions of the satellites of Jupiter and those of Saturn, we can also make use of these integrals. Temporarily stable resonance conditions are therefore more probable, relatively to the number of bodies under observation, than in the asteroid problem. And in fact, such a relation between the motions of three of the inner satellites of Jupiter is known to exist at the present time. And there is another, of a more complicated character, between two of the satellites of Saturn.

The case of Jupiter and Saturn is an interesting one in this connection, because the ratio of their periods is nearly 2:5. Another is that of Neptune and Uranus whose periods are nearly 2:1. While they are well outside the resonance "ranges," it is not possible to assert that they never have been or never will be within these ranges. Neither case permits small oscillations within the ranges. It is indeed entirely possible that they illustrate again the fact mentioned above, namely, that the positions which have the highest temporary stability are those just outside the largest resonance ranges. This fact has been pointed out as an observational result by previous workers. My argument here has been an attempt to show that these are probably not configurations of permanent stability, but merely those in which a much longer time will be spent than within the resonance regions.

A few words in reference to the application of the theory of resonance to the orbit of the moon may conclude this part of my subject. It has, owing to its proximity to the earth, received far more attention from the celestial mechanician than any other body, and its motion has been calculated to a high degree of accuracy. This calculated motion, involving at least ten arguments derived independently from observation, contains many hundreds of linear combinations of these arguments and thousands of other combinations have been examined. In no case has a combination giving a resonance with an observable coefficient been found. And yet, owing to tidal friction we know that its mean angular velocity is subject to a real secular change. On the law of chance alone we should not expect this result. If, however, the result obtained above is correct, namely, that a resonance condition with a large periodic perturbation moves the body away from the region rapidly, the chances of finding it there are much diminished. The changes in mean angular velocity due to tidal friction would therefore not be steady over periods measured by millions of revolutions but would contain intervals where small changes took place with comparative rapidity, the longer intervals of slow change being similar to that which characterizes the motion of the moon at the present time.

What general conclusions can we draw from these configurations? The first and the most obvious is that our knowledge of the past or future of the solar system must be a function of the interval within which it has been under observation, so far as the law of gravitation alone is concerned. The absence of a knowledge of any integrals of the problem of three or more bodies, beyond those known, is partly responsible for this result. If we try to represent the motion by oscillating functions we are met with the difficulty that these give an indeterminate result beyond a certain degree of accuracy. It is possibly true that there are no secular changes in the strict sense in any of the linear coordinates, but it seems equally true that, owing to resonance, we cannot lay down any limit to the amplitudes of these oscillations and thus limit the changes of the present configuration. There appears to be no regular increase in these amplitudes; it becomes rather a question of probability than of calculation as to whether any particular configuration will be fundamentally changed within a very long interval of time. It seems necessary that the smaller bodies of the solar system should have orbits which are under continuous development and this necessarily carries with it a continuous development of the whole system.

One way out of the difficulty which has been sometimes suggested is the adoption of a "quantum" frequency which would permit of complete periodicity. This frequency, however, would have to correspond to a period much longer than the probable past history of the solar system. To be logical it should extend to the whole material universe and should therefore be independent of the amount of matter present, that is, it should be a universal constant of nature—a property of space in the sense of Einstein. The assumption has philosophic interest only since it appears to have no practical bearing on the actual problem on account of the time factor involved and of other reactions of matter which are continually altering the conditions of motion.

This last remark is equally applicable to assumptions concerning the existence of periodic solutions corresponding to an arbitrary set of initial conditions; such solutions, if they exist, will have periods too long for us to be able to say that the non-gravitational forces have not had a sensible effect in the period. It does not seem possible to avoid the conclusion that small changes in the motion at any time will ultimately produce finite changes in the configuration, the amount of the change being a function of the interval, as well as of the other conditions. This amounts to the statement that the solar system, and indeed any gravitational system consisting of several bodies, is always in a state of development for gravitational reasons alone. The question is therefore shifted to that of the speed of this development, and I have attempted to show that in this speed the phenomena of resonance play a considerable part. Accumulated evidence goes to show that the history of the solar system from its formation into discrete bodies is much greater than 10^9 years. During such an interval it appears to be doubtful whether we can deduce much as to the initial configuration from the present configuration, except what the integrals of energy and angular momentum may furnish.

Certain of the present statistical distributions appear to be the results of chance in the same sense that the result of flipping a coin is said to be due to chance. In other words, a limited amount of information is to be obtained from considerations of probability. Thus the few known asteroids of the Trojan group are still there because the initial conditions of a large number presumed to have been there originally are such that the chance of still finding some there is not too small. We find few or none in the 2:1 and 3:1 ranges and perhaps in other resonance ranges because the chance of finding one under oscillation conditions is very small. The dark rings of Saturn are partly denuded of members of the system for the same reason. I am inclined to believe that the remarkable "families" of asteroid orbits deduced by K. Hirayama* will ultimately be found to be the results of perturbation effects due to resonance points with large ranges, rather than to some original configuration in which the bodies were all portions of some single body; in fact that "families" are due to the relatively stable regions just outside the resonance regions with large ranges. They are observable because the laws of chance indicate that at any one moment more asteroids will be found there than elsewhere.

It has already been pointed out that under resonance conditions, and especially those of the 2:1 ratio, the eccentricities will vary greatly. If a large number of bodies

* Japanese Journal of Astronomy, Vol. 1, No. 3.

were started near such a region, the chance of one of them getting close to Jupiter on account of large eccentricity is greatly increased. Of these a certain proportion will have their eccentricities so greatly altered as to move afterwards in cometary orbits. The periods will, in general be much less changed. On this basis, the numerous short-period comets with periods nearly half that of Jupiter would be regarded as a succeeding stage of the evolution of asteroids rather than a previous stage as suggested by Leuschner in his paper on the Pons-Winnecke Comet already referred to. The close approach to Jupiter would turn loose surface material on the asteroid into satellite orbits about the asteroid and thus perhaps furnish the material for the "tail", further loosening of the material resulting from the temperature changes taking place at each near approach to the sun. Some evidence in favor of this hypothesis is furnished by the apparent dissolutions of two or more comets which have taken place within historic times. It is difficult to account for this last fact except on the hypothesis that the supply of comets is being renewed. The same mode of development should apply to the rings of Saturn. There, however, an eccentric "stone" is liable to collide with other members of the system before it gets very close to one of the satellites, and thus a certain average circularity would be preserved.

The conclusion that the calculus of probabilities is more likely to lead to further information than the logical processes of exact analysis is perhaps a confession of defeat. For probability itself is here nothing but the chance of our being correct or incorrect in any statement, so long as we suppose that a determinate set of initial conditions will always produce a determinate result. When, however, we remember that our conditions are never exactly determinate in the mathematical sense on account of the inevitable errors of measurement, and that a very small change in the measurements will ultimately produce a finite change in the calculated configuration, this procedure is perhaps the only way in which we can deduce the needed information.

I am very fully aware of the unsatisfactory nature of an address supposed to be of a mathematical and therefore logical character which, at least partly, contains speculations as to mathematical or physical results which would seem to be capable of being logically deduced from the data. The reason for this procedure is the present condition of investigation in celestial mechanics. While a subject is rapidly advancing under the combined attacks of theory and experiment, speculations published without a sufficient background of observation are often harmful to progress because they diminish the interest of those who, equally capable of making the same speculations, take time to test them by experiment or calculation. But where a subject seems to have reached a condition where little progress is being made, and this I think is true as far as the gravitational development of the solar system is concerned, some speculation may be useful, if it indicates new avenues of approach or a more thorough exploration of old avenues. The speculations indulged in here have, however, a basis of physical and mathematical analogies and of experience gained from special examples. The processes followed have indicated how certain observed phenomena can be explained and have indicated also certain limits to the amount of information which can be obtained from the methods at present in use. The arguments lack precision; one question raised is whether it is possible to obtain precision. Our capacity to calculate and measure is limited, and it is this capacity which seems to be a part of the general question. In any case, it cannot be laid aside when we desire to obtain information as to the remote past or remote future of the solar system.

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