

THE FEBRUARY MEETING IN NEW YORK

The two hundred fifty-ninth regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, February 25, 1928, extending through the usual morning and afternoon sessions. Attendance included the following eighty-six members of the Society:

R. C. Archibald, R. G. Archibald, J. W. Arnold, Beetle, Benton, Bratton, Carrie, E. T. Carroll, Alonzo Church, Clark, Clutz, Cramlet, Dehn, Derby, Doermann, Dresden, Engstrom, Feld, Fiske, Fite, D. A. Flanders, Fort, Frink, Gale, Gehman, Gill, Glenn, Gronwall, C. C. Grove, L. S. Hill, Hille, Himwich, Hofmann, Hollcroft, Joffe, M. I. Johnson, R. A. Johnson, Kasner, Kenny, Kline, Knebelman, Koopman, J. P. Kormes, Mark Kormes, Langman, Lefschetz, Lehr, Littauer, L. T. Moore, T. W. Moore, Moyle, Mullins, F. H. Murray, C. A. Nelson, Ore, Pepper, Pfeiffer, Pixley, Post, Raudenbush, Rawles, Reddick, M. S. Rees, Robertson, Roos, Rutt, Schoonmaker, Seely, Serghiesco, Simons, Smail, Virgil Snyder, J. M. Thomas, T. Y. Thomas, Tracey, Veblen, Wedderburn, Weida, Weisner, H. S. White, Whittemore, W. M. Whyburn, Widder, W. A. Wilson, J. W. Young, Margaret M. Young.

There was no meeting of the Council or of the Trustees. President Snyder presided at the morning session, relieved in the afternoon by Vice-President Young.

Titles and abstracts of the papers read at this meeting follow below. The papers of Gergen, Ghosh, Prasad, Trjitzinsky, G. T. Whyburn, and Zarycki were read by title. Mr. Ghosh was introduced by the Secretary, and Dr. Zarycki by Professor Dunham Jackson.

1. Dr. Louis Weisner: *Polynomials $f[\phi(x)]$ reducible in fields in which $f(x)$ is irreducible.*

Given a polynomial $f(x)$ with coefficients in R and irreducible in R , the author determines all polynomials $\phi(x)$ with coefficients in R such that $f[\phi(x)]$ is reducible in R .

2. Dr. R. G. Archibald: *The impossibility of a separation of types of linear odd divisors of binary quadratic forms.*

This paper develops theorems on the existence of solutions of quadratic congruences in two unknowns and utilizes these results to prove the

impossibility of a separation of types of linear odd divisors of binary quadratic forms. Tables are given in Legendre's *Théorie des Nombres* of the quadratic divisors and of the linear odd divisors of a form $\ell^2 + au^2$ for certain integral values of a . In many cases the forms of the linear odd divisors are paired off in sets with a single quadratic divisor. This paper is concerned with the remaining cases in Legendre's Tables IV-VII in which the forms of the linear odd divisors of $\ell^2 + au^2$ are paired off in sets with more than one quadratic divisor. If the linear odd divisor $4ax + \alpha$ corresponds to two or more quadratic divisors of $\ell^2 + au^2$, the question is raised whether it is possible to find a positive integer n such that all the divisors of $\ell^2 + au^2$ contained in each of $4any + (j4a + \alpha)$, $j = 0, 1, \dots, n-1$, are represented, respectively, by one and only one of the quadratic divisors of $\ell^2 + au^2$. By the method here developed, it is found that such a separation is impossible.

3. Professor J. I. Tracey and Dr. L. T. Moore: *Covariant conditions for multiple roots of a binary form.*

This paper gives a method whereby a covariant may be formed which vanishes when an n -ic has a p -fold root, a p_1 -fold root, a p_2 -fold root, \dots , a p_j -fold root simultaneously. It also enables one to determine the other systems of equality among the roots for which the same covariant will vanish.

4. Professor O. E. Glenn: *On the generalization of the galoisian complex domain.*

For every integral modulus n there exists a form $f(x) = x^2 - x + r$, with r integral, having no integral roots modulo n . Assume an imaginary ι such that $f(\iota) \equiv 0 \pmod{n}$. Any polynomial in ι with integral coefficients is reducible by $\iota^2 = \iota - r$ to $\alpha\iota + \beta$, where α, β are integral residues of n . These n^2 complex numbers form a domain Δ in which a congruence whose coefficients are residues of n , $F(x) = a_0x^m + a_1x^{m-1} + \dots + a_m \equiv 0 \pmod{n}$, may have N roots and, if n is composite, $N <, =, \text{ or } > m$. This paper, in its present form, is concerned with necessary and sufficient matrix conditions, analogous to the algebraic theory of Stuyvaert and Dines, in order that two polynomials F may have a common root in Δ .

5. Dr. Stefan Serghiesco: *On the number of roots of a system of n equations.*

In a previous paper presented to the Society, the author introduced an integral containing certain differential invariants formed from a given system of n equations, and showed that, in a special case, this integral gives the number of common roots of the system of equations. This present paper contains a further extension of this theory, and shows that by means of the same integral we may obtain a formula giving the general solution of the problem. The formula, which is a difference of two integrals, and the method are entirely different from those used in the Picard-Kronecker theory of the same problem.

6. Dr. W. M. Whyburn (National Research Fellow):
On self-adjoint, ordinary differential equations of the fourth order.

The general, real, self-adjoint, ordinary differential equation of the fourth order and the second-order equation with coefficients that are complex functions of a real variable are studied through a consideration of the pair of real, second-order differential equations

$$\frac{d^2 y}{dx^2} + p(x)y = q(x)z, \quad \frac{d^2 z}{dx^2} + p(x)z = r(x)y$$

Simple conditions are derived under which the integral curve in the yz -plane will be essentially spiriliform and will make an arbitrarily large number of circulations about the origin as x ranges over the entire real axis. Related integral equations are studied, and used to develop comparison theorems of the type obtained by Birkhoff (Annals of Mathematics, (2), vol. 12, pp. 103-127) for third-order systems. Several other properties of the system are exhibited through their relation to similar properties of the second-order equation $u'' + p(x)u = 0$. A dynamical interpretation is used for the equations and the hypotheses employed.

7. Professor Tomlinson Fort: *A series resembling factorial series.*

The author considers series of the type

$$\sum_{n=1}^{\infty} c_n \lambda_1 \lambda_2 \cdots \lambda_n / ((z + \lambda_1)(z + \lambda_2) \cdots (z + \lambda_n)),$$

where $0 \leq \lambda_{n-1} \leq \lambda_n \rightarrow \infty$. He proves a variety of theorems concerning convergence and summability. The series in question is seen to bear a relation to the factorial series,

$$\sum_{n=1}^{\infty} c_n n! / (z(z+1) \cdots (z+n)),$$

resembling that borne by the Dirichlet series, $\sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$, to the special Dirichlet series $\sum_{n=1}^{\infty} c_n n^{-z}$.

8. Professor D. V. Widder: *The singularities of functions defined by the Stieltjes integral $\int_0^{\infty} e^{-st} d\alpha(t)$.*

Seeking to generalize power series and Dirichlet series by replacing the discrete set of numbers λ_n that appears in such series, $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, by a continuous set, the author considers a Stieltjes integral of the form $f(s) = \int_0^{\infty} e^{-st} d\alpha(t)$, where $\alpha(t)$ is a function of bounded variation in every finite interval, $0 \leq t \leq R$. Functions of this form, in which $\alpha(t)$ has a continuous derivative, were first studied by Laplace, the function $\alpha'(t)$ having been designated as the determining function, $f(s)$ as the generating function. The present paper undertakes, for the first time, the study of the singularities of such functions, as determined by the generating function. The most important result states that if the function $f(s)$ has singularities at points a , and if the function $\phi(s) = \int_0^{\infty} e^{-st} d\beta(t)$ has singularities at points b , then $F(s) = \int_0^{\infty} e^{-st} \alpha(t) d\beta(t)$ has singularities at most at points $a+b$ and b , under certain conditions imposed on the rates of increase of $f(s)$ and $\phi(s)$.

on the vertical lines of the complex plane. From further developments it is found that the well known theorems of Hadamard, Hurwitz, and Faber, three of the most fundamental theorems of analytic continuation, appear as special cases under the present theory.

9. Dr. T. H. Rawles: *Two classes of periodic orbits with repelling forces.*

The purpose of this paper is to obtain certain classes of periodic orbits of a system consisting of one body of very great mass which attracts two mutually repellant bodies of very small mass. It will be shown that if certain conditions of symmetry are imposed the orbits of the two small bodies must lie in two parallel planes or else be coplanar. Expressions for these two types will be obtained.

10. Professor W. A. Wilson: *Some properties of upper semi-continuous collections of continua.*

Let $M = \Sigma[X]$ denote the sum of a set of disjoint bounded continua in a plane, let t denote any point of a circumference C , and let there be a $(1, 1)$ correspondence between the elements of M and the points of C such that $X = f(t)$ is upper semi-continuous in C . R. L. Moore has shown that if no element X separates the plane, then M divides the plane into two domains whose frontiers are parts of M . By imposing the condition that $X = f(t)$ is a *minimal* upper semi-continuous function, that is, that there is no function $Y = g(t)$ defined over C such that each Y is a sub-continuum of X and some $Y \neq X$, it is shown that M is the frontier of both its complementary domains. If, in addition, the condition that the elements X do not separate the plane is removed, M is still the common frontier of two complementary domains and the frontier of each other complementary domain is a part of some element. Finally, if $f(t)$ is a minimal upper semi-continuous function defined over a segment $a \leq t \leq b$, and no element separates the elements $X_a = f(a)$ and $X_b = f(b)$, M is a continuum irreducible between the sets X_a and X_b .

11. Mr. J. J. Gergen: *On accessible points on the boundary of a three-dimensional region.*

The author is engaged in generalizing to three dimensions some of the results of W. F. Osgood and E. H. Taylor established in *Conformal transformations on the boundaries of their regions of definition* (Transactions of this Society, vol. 14 (1913), p. 277). The chief result of the present paper is that an accessible point on the boundary of a simply connected three-dimensional open region, on which the Green's function vanishes, corresponds at most to a point set of zero measure on the unit sphere, when the interior of the region is transformed into the interior of the unit sphere by means of the correspondence set up by the Green's functions and their orthogonal trajectories. Examples can be given to show that this is the most general result that can be obtained merely in these terms,

without restricting the character of the boundary point beyond mere accessibility.

12. Professor Ganesh Prasad: *On the nature of θ in the mean-value theorem of the differential calculus.*

If $f(x)$ is a single-valued function which is finite and continuous in an interval (a, b) the ends being included, then the relation $f(x+h) - f(x) = hf'(x+\theta h)$, $0 < \theta < 1$, holds for every value of x and h for which the interval $(x, x+h)$ is in the interval (a, b) , provided that either $f'(x)$ exists for every internal point of (a, b) or a certain less restrictive condition is satisfied. The object of the present paper is to consider θ as a function of h , x being taken a constant, say 0, and to show (1) that θ is not necessarily single-valued, (2) that, even if we fix upon a special group of values of θ , which may be called a single-valued function of h , such a function is not necessarily continuous, and (3) that, even if θ is continuous, it is not necessarily differentiable.

13. Mr. Suddhodan Ghosh: *On the solution of the equations of elastic equilibrium suitable for elliptic boundaries.*

A solution in elliptic coordinates of the two-dimensional problem of equilibrium of an elastic isotropic body in the absence of body forces is given in the present paper, and applied to two problems of plane strain. A single-valued solution is found, suitable for the case of an elliptic cylindrical cavity in an infinite solid, and also a many-valued solution applicable to the state of strain in an elliptic cylindrical shell which has suffered dislocation due to a triangular axial fissure. Love has given an outline of a method for solving problems in plane strain in elliptic coordinates, but his method is only applicable to cases where the surface displacements are given; it is shown that the method developed in this paper lends itself easily to the construction of the solution of the particular problem dealt with by Love.

14. Professor W. J. Trjitzinsky: *Functions with assigned initial values.*

The problem of constructing indefinitely differentiable functions of a real variable with assigned initial-values at a point has been undertaken by de la Vallée Poussin in connection with the study of quasi-analytic functions; he gives the representation in the form $\sum \alpha_j \cos m_j x$. In the present paper the problem is treated more completely; representations are given (1) in a series of power series, (2) a series of infinite integrals, and (3) with assigned initial values at a singular point, which is exhibited as a limiting point of a sequence of poles. An asymptotic representation is given in connection with the second method; also, applying a formula of Darboux, a more general asymptotic expansion is derived.

15. Professor G. T. Whyburn: *Concerning the cuttings of a continuous curve.*

Let M be any bounded plane continuous curve, and if R is an open subset of M , let $F(R)$ denote the boundary of R with respect to M . In this paper the following results are proved. (1) There exist in M , an irreducible cutting K of M , and a component H of K such that no point of H is accessible from any component of $M-K$. (2) If R is any connected open subset of M , then each component H of $F_m(R)$ is "accessible" from R , that is, if A is any point of R , there exists a continuum N containing both A and H , which is a subset of $R+H$. (3) If K is an irreducible cutting of M between two points A and B of M , and R_c is any component of $M-K$ which contains neither A nor B , then $F_m(R_c)$ contains points of not more than two components of K . (4) Suppose K is an irreducible cutting of M which itself has at least two components and is such that $M-K$ has at least three components. Then if any component of K is decomposable, K consists of exactly two points.

16. Dr. H. P. Robertson (National Research Fellow):
Invariants of contact transformations.

A tensor calculus applicable to contact transformations is developed, and by means of it a theory of the invariants of such transformations constructed. The Hamiltonian theory of dynamics is identified with the invariant theory of a single scalar.

17. Professor Einar Hille: *Note on the behavior of certain power series on the circle of convergence with application to a problem of Carleman.*

The paper contains a study of the properties of convergence of power series of the form $\sum f(n) \exp [2i a(n)] z^n$, where $f(n)$ and $a(n)$ are subjected to certain restrictions. There exist power series of this type which are continuous in and on the unit circle and are such that $\sum [f(n)]^r$ converges for no $r < 2$. This paper will appear in an early issue of the Proceedings of the National Academy.

18. Professor Solomon Lefschetz: *Duality relations in analysis situs.*

Given two complexes C_n and $C_k \subset C_n$, where C_k includes the boundary F of C_n , there may be defined certain simultaneous invariants of the pair which generalize the Betti numbers. These numbers are shown to satisfy duality relations that include directly or indirectly all those now known in analysis situs, notably the relations apparently wholly unrelated due to Poincaré and to Alexander. The same relations are extended to the case of any closed set $G \subset C_n$, with $F \subset G$.

19. Dr. Miron Zarycki: *General properties of Cantorian coherences.*

The author develops on a postulational basis that part of point set theory which concerns coherences and adherences. The postulates are

those underlying the usual algebra of logic and three others which become familiar theorems of point set theory when the undefined terms are given a particular interpretation. These three postulates are shown to form an absolutely independent set.

20. Professor J. R. Kline: *Concerning the dimension of a continuous curve every subcontinuum of which is a continuous curve.*

The author proves that every continuous curve, which has the property that every one of its subcontinua is a continuous curve, must be one-dimensional in the sense of Menger and Urysohn.

21. Professor Edward Kasner: *General theory of element-point transformations.*

The author studies the general transformation T from a lineal element (x, y, y') of a first plane I to a point (X, Y) of a second plane II. To a point of I corresponds a curve of II. To a point of II corresponds a series of elements in I. By eliminating y' from $X = X(x, y, y')$, $Y = Y(x, y, y')$, we obtain a related contact transformation Γ . To a curve in I corresponds by T a curve in II, but to a curve in II corresponds a field of elements and therefore ∞^1 curves in I. Special types of transformation are studied, and in particular the type arising in the author's theory of polygenic functions.

22. Professor Edward Kasner: *Additive groups of transformations.*

The author considers sets of point transformations which form a group in a sense different from the usual sense. Instead of combining two transformations T_1, T_2 by the usual multiplication T_1T_2 , he considers vectorial addition. If T_1 transforms the point P into P_1 , and T_2 transforms P into P_2 , then the sum of the vectors PP_1 and PP_2 gives PP_3 and the transformation from P to P_3 is defined as $T_1 + T_2$. For example, affine transformations form a group in this sense, but projective transformations do not. Attention was called to additive groups incidentally in connection with Darboux transformations by the author in 1909 (see Transactions of this Society, vol. 15, p. 212, last foot note). Many new examples are now given. In particular the set of all harmonic transformations $X = \phi(x, y)$, $Y = \psi(x, y)$, where ϕ and ψ obey the Laplace equation. (See Proceedings of the National Academy, vol. 14 (1928), pp. 75-82, last paragraph.)

23. Dr. Lulu Hofmann and Professor Edward Kasner: *Homographic circles.*

This paper will appear in full in an early issue of this Bulletin.

24. Professor Edward Kasner: *The increment ratio of any two polygenic (or non-analytic) functions.*

In earlier papers (see *Science*, vol. 66 (1927), p. 581 and abstracts in this *Bulletin*), the author studied the derivative dw/dz , where $w = \phi(x, y) + i\psi(x, y)$, the components ϕ and ψ being arbitrary. In the present paper he studies dw_1/dw_2 , where w_1 and w_2 are any such polygenic functions. To each point $z = x + iy$ corresponds a circle in the derivative plane, as in the simpler case, but the correspondence between the directions $m = dy/dx$ and the points of the circle is now a general homographic correspondence instead of being uniform (rate $-2:1$). The only cases in which dw_1/dw_2 is uniform are when w_2 is an analytic function of $x + iy$ or of $x - iy$, in the latter case the rate being $(2:1)$. If all the derivative circles reduce to points, w_1 must be an analytic function of w_2 . To all the points of the z plane corresponds a congruence of homographic clocks. If a congruence is given arbitrarily, necessary and sufficient conditions are obtained for its being identifiable with a derivative congruence.

25. Professor T. R. Hollcroft: *The lines of an algebraic surface*.

In this paper, the number of invariants necessary and sufficient for a surface to contain a line or a given system of lines is found. From this is obtained the maximum number of independent lines or lines of a given system lying on an algebraic surface of given order. These limits hold when the surface is otherwise non-singular. When the surface has multiple points, lines, or curves, and these are accompanied by sets of lines on the surface, the lines of such sets do not account for additional invariants. The monoid is an example of this.

ARNOLD DRESDEN

Associate Secretary

A CORRECTION

On page 139 of the March-April issue (vol. 34), the last sentence in the abstract of paper No. 17 is incorrect and should have been omitted.