EXTENDED POLYGONAL NUMBERS*

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1. *Introduction and Summary.* The xth extended polygonal number of order $m+2$ is $e(x)=\frac{1}{2}m(x^2+x)-x$, while $e(-x)$ is the xth polygonal number of order $m+2$. We take $m>2$ and thereby exclude the classic cases of triangular numbers and squares. If k is an integer $\geqq 0$, the values of

$$(1) \qquad e(x-k) = (x-k)\left[\tfrac{1}{2}m(x-k+1)-1\right]$$

for integers $x\geqq 0$ are the first k polygonal numbers and all the extended polygonal numbers.

We shall prove that every integer $p\geqq 0$ is a sum of E numbers 0 or 1 and four values of function (1) for integers $x\geqq 0$, where $E=m-2$ if $k=0$; $E=m-3$ if $k=1$; while if $k=2$, $E=m-6$ when $m\geqq 8$, $E=2$ when $m=7$, $E=1$ when $m=5$ or 6, $E=0$ if $m\leqq 4$; and finally if $k\geqq 3$, $E=m-6$ when $m\geqq 7$, $E=1$ when $m=6$, and $E=0$ when $m\leqq 5$. In no case will a smaller E serve for every p.

These results have a very simple interpretation when $E=0$, so that every integer $p\geqq 0$ is a sum of four values of (1). When $k=1$, $m=3$, this is equivalent to the fact that every positive integer $24p+4$ is the sum of the squares of four numbers of the form $6x-5$ with $x\geqq 0$. When $k=2$, $m=4$, and when $k=3$, $m=5$, the equivalent facts are

$$8p+4=\sum(4x-7)^2, \quad 40p+36=\sum(10x-27)^2,$$

each summed for four integers $x\geqq 0$. When $k=1$, $m=4$, the equivalent fact is that, for every integer $p\geqq 0$, one of $8p\pm 4$ is a sum of the squares of four numbers $4x-3$ with $x\geqq 0$. Each of our theorems has a similar interpretation concerning four squares, besides the obvious one for $E+4$ squares. The single improvement on the last fact is furnished by the last part of Theorem 3, which is equivalent to

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$$56A + 125 = \sum_5 (14x - 23)^2,$$

for five integers $x \geqq 0$ and every $A \geqq 0$. By Theorem 5,

$$3p + 4 = \sum_4 (3x - 8)^2 \quad \text{if} \quad p \neq 8n + 4.$$

Assistance was provided by the Carnegie Institution for preparing the tables and verifying the theorems for the necessary initial cases. The tables in the three papers are of constant use in a memoir to appear in the American Journal of Mathematics, which treats the like problem for all quadratic functions.

2. *General Formulas.* A quadratic function has an integral value for every integer $x \geqq 0$ if and only if it has the form $\frac{1}{2}mx^2 + \frac{1}{2}nx + c$, where m, n, c are integers such that $m+n$ is even. Comparison with (1) gives

$$(2) \quad n = m - 2 - 2mk, \qquad c = \tfrac{1}{2}m(k^2 - k) + k.$$

For these values, formulas (4), (6), (8), (13), and (15) of the writer's paper in this Bulletin (vol. 33 (1927), pp. 713–720) give

$$(3) \quad A = mw + 4c + r - b, \qquad w = \tfrac{1}{2}(a + b) - kb,$$

$$(4) \quad U = 24mA + m^2(9 - 12k - 12k^2) - 24mk - 60m + 36,$$

$$(5) \quad V = 2mA - 2mE + (m - 2)^2, \quad P = 2m(d - k) - m - 2,$$

$$(6) \quad F = (2V + W)^2 - VU > 0, \quad 8W = U - P^2.$$

The (reduced) minor conditions are

$$(7) \qquad A \geqq 4c + 4E, \qquad A \geqq 4c + \tfrac{2}{3}m$$

if $n \geqq 0$, but the same and

$$(8) \qquad 3A \geqq 12c + 2m - 2n - n^2/m, \qquad (\text{if } n < 0).$$

When $k \geqq 2$, the latter follows from (7_2). In fact, the sum of its last two terms is negative, since $-n$ is positive and $2m+n$ is negative.

3. *The Case* $k=0$. We shall prove that every integer

$A \geqq 0$ is a sum of $E = m-2$ numbers 0 or 1 and four extended polygonal numbers $0,\ m-1,\ 3m-2, \cdots$. Note that $E(m-2) = m-2$. When r takes the values $0, 1, \cdots, m-2$, and b the odd values β and $\beta-2$, evidently $b-r$ takes the values of $\beta, \beta-1, \cdots, \beta-m$, which include a complete set of residues modulo m. We choose $b-r$ congruent to $4c-A$. Then (3) determines an integer w and hence an odd integer a. There will be two consecutive odd values β, $\beta-2$ of b if the difference between the limits for b exceeds 4. Hence we take $d=4$. Then

$$U = 24mA + 9m^2 - 60m + 36, \qquad V = 2mA - m^2 + 4,$$

$$P = 7m - 2, \qquad W = 3mA - 5m^2 - 4m + 4,$$

$$F = m^2A^2 - 92m^3A + 64m^2A + 58m^4 - 4m^3 - 152m^2 + 144m > 0,$$

which evidently holds if $A \geqq 92m-64$, $m \geqq 3$. Conditions (7) hold if $A \geqq 4m$.

In Table III*, the entries involving the same multiple of m and the intervening numbers will be said to form a *block*. We suppress $15m-5, \cdots, 69m-13,\ 14, \cdots$, and all such entries down to the last entry of any block which differs by 3 or more from the next entry of that block. If we subtract the largest entry of any abridged block from the least entry of the next abridged block, we always obtain a difference $\leqq m-1$.

TABLE III. SUMS OF FOUR EXTENDED POLYGONAL NUMBERS

$0,\ m-1,\ 2m-2,\ 3m-2$–$3,\ 4m-3$–$4,\ 5m-4,\ 6m-3$–$5,\ 7m-4$–$5,\ 8m-5$–$6,\ 9m-5$–$6,\ 10m-4,\ 6,\ 7,\ 11m-5,\ 7,\ 12m-6$–$8,\ 13m-6$–$8,\ 14m-7,\ 8,\ 15m-5,\ 8,\ 9,\ 16m-6$–$9,\ 17m-7$–$9,\ 18m-7$–$10,\ 19m-8$–$10,\ 20m-8$–$10,\ 21m-6,\ 8,\ 9,\ 11,\ 22m-7,\ 9$–$11,\ 23m-8,\ 10,\ 11,\ 24m-8$–$12,\ 25m-9,\ 11,\ 12,\ 26m-10$–$12,\ 27m-9$–$12,\ 28m-7,\ 10$–$13,\ 29m-8,\ 11$–$13,\ 30m-9$–$13,\ 31m-9$–$13,\ 32m-10$–$14,\ 33m-11$–$14,\ 34m-10$–$14,\ 35m-11$–$13,\ 36m-8,\ 11$–$15,\ 37m-9,\ 12$–$15,$

* Tables No. I and No. II occur in the previous papers by the author, this Bulletin, vol. 33, pp. 713-720, and vol. 34, pp. 63-72.

$38m-10$, 11, 13–15, $39m-10$–13, 15, $40m-11$, 13–16, $41m-12$–16, $42m-11$, 12, 14–16, $43m-12$, 13, 15, 16, $44m-13$–16, $45m-9$, 13–17, $46m-10$, 12, 14–17, $47m-11$, 13, 15–17, $48m-11$, 12, 14–17, $49m-12$–17, $50m-13$–18, $51m-12$, 13, 15–18, $52m-13$–18, $53m-14$–17, $54m-14$, 15, 17, 18, $55m-10$, 13, 15–19, $56m-11$, 14, 16–19, $57m-12$, 14–17, 19, $58m-12$, 13, 15–19, $59m-13$, 16–19, $60m-14$, 16–20, $61m-13$–17, 19, 20, $62m-14$–20, $63m-15$–20, $64m-15$–20, $65m-14$, 16, 17, 19, 20, $66m-11$, 15, 17–21, $67m-12$, 16–21, $68m-13$, 16–21, $69m-13$, 14, 17, 19–21, $70m-14$, 15, 18–21, $71m-15$–21, $72m-14$–22, $73m-15$–17, 19–22, $74m-16$–19, 21, 22, $75m-16$–22, $76m-15$–22, $77m-16$, 17, 19–22, $78m-12$, 17–23, $79m-13$, 17–21, 23, $80m-14$, 18–23, $81m-14$–17, 19–23, $82m-15$, 17–23, $83m-16$–21, 23, $84m-15$, 16, 18–24, $85m-16$, 17, 19–24, $86m-17$, 19–24, $87m-17$–21, 23, 24, $88m-16$, 18, 20–24, $89m-17$, 19–24, $90m-18$, 19, 21–25, $91m-13$, 18–21, 23–25, $92m-14$, 19–25.

THEOREM 1*. *If* $k=0$, *then* $E=m-2$.

4. *The Case* $k=1$. The following is a complete list to $8m-4$ of sums by four of 1 and extended polygonal numbers:

$$
\begin{aligned}
&0-4,\ m-1,\ m,\ m+1,\ m+2,\ 2m-2-1,\ 2m,\\
&3m-3-1,\ 3m,\ 3m+1,\\
&4m-4-1,\ 5m-4-3,\ 6m-5-1,\ 6m,\\
&7m-5-2,\ 8m-6-4.
\end{aligned}
\tag{9}
$$

Hence $E(6m-6)=m-3$. We next prove

THEOREM 2. *If* $k=1$, *then* $E=m-3$.

First, let $m\geqq 4$. For $b=\beta$, $\beta-2$ and $r=0, 1, \cdots, m-3$, then $b-r$ takes the values β, $\beta-1, \cdots, \beta-(m-1)$, which form a complete set of residues modulo m. Hence $d=4$. Then

* In the current number of the Proceedings of the American Philosophical Society, the writer gives another proof, analogous to that by Cauchy for ordinary polygonal numbers.

$$U = 24mA - 15m^2 - 84m + 36, \quad V = 2mA - m^2 + 2m + 4$$
$$P = 5m - 2, \quad W = 3mA - 5m^2 - 8m + 4,$$
$$F = m^2A^2 + m^2A(64 - 44m) + 34m^4 + 2m^3 + 112m^2 + 168m.$$

Evidently $F>0$ if $A \geqq 44m-64$. The minor conditions hold if $A \geqq 4m$. By (9), $E(A) \leqq m-3$ for $A \leqq 8m-4$. We annex $15m-7=1+3m-2+2(6m-3)$ to Table III and abridge it as in §3. Then the gaps are $\leqq m-2$ from $8m-4$ to $44m-16$. This proves Theorem 2 when $m \geqq 4$.

Second, let $m=3$. Then β, $\beta-2$, $\beta-4$ form a complete set of residues modulo 3. Hence $d=6$,

$$U = 72A - 351, \quad V = 6A + 1, \quad W = 9A - 122,$$
$$F = A^2 - 334A + 1639 > 0 \quad \text{if} \quad A \geqq 6.$$

The minor conditions hold if $A \geqq 7$. But the numbers $<3m$ in (9) are 0-8. Hence Theorem 2 holds if $m=3$.

5. *The Case* $k=2$. Then $E(m-2)=m-6$.

First, let $m \geqq 8$. We shall prove that $E=m-6$. For $b=\beta$, $\beta-2$, $\beta-4$, $\beta-6$, $0 \leqq r \leqq m-6$, evidently $b-r$ takes the values β, $\beta-1, \cdots, \beta-m$, whence $d=8$. Then

$$U = 24mA - 63m^2 - 108m + 36, \quad V = 2mA - m^2 + 8m + 4,$$
$$P = 11m - 2, \quad W = 3mA - 23m^2 - 8m + 4,$$
$$F = m^2A^2 - 200m^3A + 136m^2A + 562m^4 - 4m^3 + 616m^2 + 336m.$$

Thus $F>0$ if $A \geqq 200m-136$ and in fact if $A=198m-136+t$, $t \geqq 0$, since then $F=t^2m^2+tm^2(196m-136)+166m^4+268m^3+616m^2+336m$.

From Table IV we suppress $4m+8$, 5, and all such entries in any block down to the last entry which differs by 3 or more from the next entry of that block. We shall prove that $E(A) \leqq m-6$ if A lies between consecutive blocks. This will follow if proved when $A+1$ is the least entry $tm-s$ of an abridged block. Then A is the sum of $m-6$ or $m-7$ and a number n in the abridged table if $tm-s$ is the sum of $m-5$ or $m-6$ and n. In other words, if $-s$ is the term free of m in the least entry of an abridged block, then $-(s-5)$ or $-(s-6)$ is that of some entry of the preceding abridged block. An inspection of Table IV shows that this holds with $-(s-5)$ except for

(10) $t = 10, 38, 56, 71, 78, 82, 91, 108, 131, 138, 142, 159, 169, 172, 176, 191.$

It holds with $-(s-6)$ for all cases in (10) except $t=10$. But for $m \geqq 8$, $10m-8$ is the sum of $m-8$ and $9m$, while the gaps of 3 from $9m$ to $9m \pm 3$ are now permissible since $E \geqq 2$.

TABLE IV. SUMS BY FOUR OF 1, $m+2$, AND EXTENDED POLYGONAL NUMBERS

0–4, $m+5$, 4, 3, 2, 1, 0, -1, $2m+6$, 5, 4, 3, 2, 1, 0, -1, -2, $3m+7$, 6, 4, 3, 1, 0, $-1-3$, $4m+8$, 5, 2, 1, 0, $-1-4$, $5m+3$, 2, 0, -1, -3, -4, $6m+4$, 1, 0, $-1-5$, $7m+1$, 0, $-1-5$, $8m+2$, 1, 0, $-1-6$, $9m+3$, 0, $-3-6$, $10m-1$–7, $11m+0$, $-1-5$, -7, $12m+1$, 0, $-2-8$, $13m+2$, -1, $-3-8$, $14m-2-8$, $15m-2-5$, $-7-9$, $16m-1-9$, $17m+0$, -1, $-3-9$, $18m+1$, -2, -3, $-5-10$, $19m-4$, -5, $-7-10$, $20m-3$, $-6-10$, $21m-3-9$, -11, $22m-2-11$, $23m-1$, -2, -4, -5, $-7-11$, $24m+0$, -3, $-6-12$, $25m-5-9$, -11, -12, $26m-4$, -6, -7, $-9-12$, $27m-5$, $-7-12$, $28m-4-13$, $29m-3-9$, $-11-13$, $30m-2$, -3, -5, -6, $-8-13$, $31m-1$, 4*, 7–13, $32m-6$–14, $33m-5$, 6, 8, 9, 11–14, $34m-8$-14, $35m-7$–13, $36m-5$–15, $37m-4$–9, 11–15, $38m-3$, 4, 6, 7, 9–11, 13–15, $39m-2$, 5, 8–13, 15, $40m-7$, 8, 10–16, $41m-6$, 9, 11–16, $42m-9$–16, $43m-8$–13, 15, 16, $44m-7$–16, $45m-6$–9, 11–17, $46m-5$–17, $47m-4$, 5, 7–13, 15–17, $48m-3$, 6, 8–17, $49m-8$, 9, 11–17, $50m-7$, 10–18, $51m-9$–13, 15–18, $52m-9$–18, $53m-8$, 9, 11–17, $54m-13$–15, 17, 18, $55m-7$–13, 15–19, $56m-6$–19, $57m-5$, 6, 8, 9, 11–17, 19, $58m-4$, 7, 10–19, $59m-9$, 10, 12, 13, 15–19, $60m-8$, 11–20, $61m-11$–17, 19, 20, $62m-10$–20, $63m-9$, 12, 15–20, $64m-13$–20, $65m-12$–17, 19, 20, $66m-8$–21, $67m-7$–13, 15–21, $68m-6$, 7, 9–21, $69m-5$, 8, 11–14, 16–21, $70m-10$, 11, 13–15, 17–21,

* From here on, we omit minus signs in continuations.

$71m-9$, 12, 13, 15–21, $72m-11$–22, $73m-11$–17, 19–22, $74m-10$, 12–19, 21, 22, $75m-12$, 15–22, $76m-13$–22, $77m-12$–17, 19–22, $78m-9$–12, 14–23, $79m-8$–13, 15–21, 23, $80m-7$, 8, 10, 11, 13–23, $81m-6$, 9, 12–17, 19–23, $82m-11$–23, $83m-10$, 12, 13, 15–21, 23, $84m-13$–24, $85m-12$–17, 19–24, $86m-11$, 14, 17–24, $87m-15$–21, 23, 24, $88m-14$–24, $89m-13$, 14, 16, 17, 19–24, $90m-12$, 15–19, 21–25, $91m-10$–13, 15–21, 23–25, $92m-9$–25, $93m-8$, 9, 11, 12, 14–17, 19–25, $94m-7$, 10, 13–25, $95m-12$, 13, 15–21, 23–25, $96m-11$, 14, 17–25, $97m-14$–17, 19–26, $98m-13$–23, 25, 26, $99m-12$, 15–21, 23–26, $100m-15$–26, $101m-14$–17, 19–25, $102m-14$, 15, 17–26, $103m-13$, 16–21, 23–26, $104m-15$, 17–26, $105m-11$–14, 17, 20–25, 27, $106m-10$–19, 21–27, $107m-9$, 10, 12, 13, 15–21, 23–27, $108m-8$, 11, 14–18, 20–27, $109m-13$, 14, 16, 17, 19–25, 27, $110m-12$, 15, 18–27, $111m-15$–21, 23–27, $112m-14$–28, $113m-13$, 16, 17, 19–25, 27, 28, $114m-15$, 18–23, 25–28, $115m-16$–18, 20, 21, 23–28, $116m-15$, 16, 18, 19, 21–28, $117m-14$, 17, 19–25, 27, 28, $118m-18$–28, $119m-18$–21, 23–28, $120m-12$–15, 17–29, $121m-11$–17, 19–25, 27–29, $122m-10$, 11, 13–19, 21–29, $123m-9$, 12, 15–21, 23–29, $124m-14$, 15, 17–19, 21–29, $125m-13$, 16, 17, 19–25, 27–29, $126m-16$–27, 29, $127m-15$–21, 23–29, $128m-14$, 16–30, $129m-17$, 19–25, 27–30, $130m-17$–23, 25–30, $131m-16$, 17, 19–21, 24–29, $132m-15$, 18, 20–30, $133m-19$–25, 27–30, $134m-18$–30, $135m-17$–21, 23–29, $136m-13$–18, 20–31, $137m-12$–17, 19–25, 27–31, $138m-11$, 12, 14, 15, 17, 18, 21–31, $139m-10$, 13, 16–20, 23–29, 31, $140m-15$, 16, 18, 19, 22, 23, 25–31, $141m-14$, 17, 19–25, 27–31, $142m-17$–27, 29–31, $143m-16$–21, 23–29, 31, $144m-15$, 18, 21–32, $145m-20$–25, 27–32, $146m-18$–23, 25–32, $147m-17$, 18, 20, 21, 23–29, 31, 32, $148m-16$, 19–32, $149m-19$–25, 27–32, $150m-18$, 20–32, $151m-19$–21, 23–29, 31, 32, $152m-18$, 19, 21–32, $153m-14$–17, 20, 21, 23–25, 27–33, $154m-13$–18, 22–33, $155m-12$, 13, 15, 16, 18, 19, 21, 23–29, 31–33, $156m-11$, 14, 17–33, $157m-16$, 17, 19–25, 27–33, $158m-15$, 18–27, 29–31, 33, $159m-18$–21, 23–29, 31–33, $160m-17$–33,

$161m-16$, 19, 21, 22, 24, 25, 27–33, $162m-20$–23, 25–34, $163m-19$–21, 23–29, 31–34, $164m-18$, 19, 21–34, $165m-17$, 20–24, 27–33, $166m-19$, 21–34, $167m-20$, 21, 23–29, 31–34, $168m-20$–27, 29–34, $169m-19$, 20, 22–25, 27–33, $170m-18$, 21–34, $171m-15$–18, 21, 23–29, 31–35, $172m-14$–19, 22–35, $173m-13$, 14, 16, 17, 19–22, 24, 25, 27–33, 35, $174m-12$, 15, 18–27, 29–31, 33–35, $175m-17$, 18, 20, 21, 23–29, 31–35, $176m-16$, 19, 22–35, $177m-19$–22, 24, 25, 27–33, 35, $178m-18$–23, 25–35, $179m-17$, 20, 23–29, 31–35, $180m-22$–36, $181m-20$–25, 27–33, 35, 36, $182m-19$–36, $183m-18$, 20, 21, 23–29, 31–36, $184m-22$–36, $185m-22$, 24, 25, 27–33, 35, 36, $186m-21$–36, $187m-20$, 21, 23–29, 31–36, $188m-19$, 22, 23, 25–36, $189m-23$–25, 27–33, 35, 36, $190m-16$–19, 22, 23, 25–27, 29–31, 33–37, $191m-15$–21, 24–29, 31–37, $192m-14$, 15, 17, 18, 20–37, $193m-13$, 16, 19–22, 24, 25, 27–33, 35–37, $194m-18$–23, 26–35, 37, $195m-17$, 20, 23, 25–29, 31–37, $196m-20$–37, $197m-19$–25, 27–33, 35–37, $198m-18$, 21, 23–37.

If $m=7$, Table IV lacks $61=9m-2$ and $62=9m-1$, since the maximum in the preceding block is $8m+2=58$ and the minimum in the subsequent block is $10m-7=63$. If $m=6$, Table IV lacks $28=4m+4=5m-2=6m-8$. If $m=5$, it lacks $23=3m+8=4m+3=5m-2=6m-7$. Hence E cannot have smaller values than those in

THEOREM 3. *If* $k=2$, $E=m-6$ *for* $m\geqq 8$, $E=2$ *for* $m=7$, $E=1$ *for* $m=5$ *or* 6, $E=0$ *for* $m=4$. *When** $m=7$, $E(A)\leqq 1$ *if* $A\neq 62$.

Since $E\leqq m-2$ for every m, conditions (7) hold if $A\geqq 8m$. This completes the proof of Theorem 3 when $m\geqq 8$. For $m=5$, 6, or 7, the values of $b-r$ for $b=\beta$, $\beta-2$, $\beta-4$ include a complete set of residues modulo m, whence $d=6$. We have the same U as before and

$$P = 7m - 2, \qquad W = 3mA - 14m^2 - 10m + 4.$$

* Hence every integer $\geqq 0$ is a sum of five values of $e(x-2)$ for $m=7$.

If $m=7$, $U=168A-3807$, $V=14A-3$, $W=21A-752$,

$$F = 49A^2 - 7\cdot 2926A + 7\cdot 80449 > 0 \quad \text{if} \quad A \geqq 389.$$

The discussion made when $m\geqq 8$ applies when $m=7$ except for $10m-8$. The latter now equals $9m-1$ and gives no trouble.

If $m=6$, $U=36(4A-80)$, $V=12A+4$, $W=18A-560$,

$$F/36 = (7A - 92)^2 - (4A - 80)(12A + 4) = A^2 - 344A + 8784,$$

and $F>0$ if $A\geqq 317$. When $A<317=55m-13$, we have only the first two cases in (10). But $38m-16=37m-10$ is in the preceding block, while $10m-8=9m-2$ exceeds by unity the greatest entry in its preceding abridged block.

If $m=5$,

$$U = 120A - 2079,\ V = 10A - 1,\ W = 15A - 396,$$

$$F/25 = A^2 - 278A + 6253 > 0 \quad \text{if} \quad A \geqq 254.$$

But $E(A)\leqq 1$ for $A<254$.

Finally, if $m=4$, we shall prove that $E=0$. Here (1) is $2x^2-7x+6$, whence $A=2a-7b+24$.

First, let A be odd. Take $b\equiv A-2 \pmod 4$. Then a and b are odd and the general method applies. Here $d=4$, $P=10$,

$$U = 4(24A - 351), \quad V = 4(2A + 1), \quad W = 12A - 188,$$

$$F/16 = (7A - 45)^2 - (2A + 1)(24A - 351) = (A - 24)^2 + 1800.$$

Second, let $A=2S$. Take $b=2B$, $B\equiv S-2 \pmod 4$. Then $a=S+7B-12$, $a\equiv 2 \pmod 4$. Hence we may apply Lemma 3 of the writer's paper in this Bulletin for January (vol. 34, pp. 63–72). Its conditions

$$4B^2 + 2B + 1 > 3a, \quad B^2 < a$$

become on elimination of a

$$8B > 19 + u^{1/2},\ 2B < 7 + v^{1/2},\ u = 48S - 231,\ v = 4S + 1.$$

The difference between these two limits for B shall exceed 4.

Hence $4v^{1/2}-u^{1/2}>23$. The left member is $\geqq 0$. Squaring twice, we get

$$S^2 - 432S + 2220 > 0, \quad S \geqq 427, \quad A \geqq 854.$$

It was verified that $E(A)=0$ for $A<854$.

6. THEOREM 4. *For $k\geqq 3$, $E=m-6$ if $m\geqq 7$, $E=1$ if $m=6$, $E=0$ if $m=5$.*

Since 0 and 1 are the only polygonal and extended polygonal numbers $<m-2$, $E(m-2)=m-6$ if $m\geqq 6$. If $m\geqq 8$, our theorem now follows Theorem 3.

When $m=7$, it remains to prove that $E=1$ if $k=3$. As at the beginning of §5, $d=8$. Then $P=61$ and

$$U = 168A - 7503, \quad V = 14A + 11, \quad W = 21A - 1403,$$
$$F/49 = A^2 - 656A + 40606 > 0 \quad \text{if} \quad A \geqq 587 = 86m - 15.$$

Conditions (7) hold if $A\geqq 101$. When $A<587$, the discussion at the beginning of §5 shows that $E(A)\leqq 1$ except for

$$10m - 8 = 9m - 1 = 2(3m - 2) + 3m + 3.$$

Let $m=5$, $k=3$. We shall prove that $E=0$. Here $d=10$, $P=63$,

$$U = 120A - 3999, \quad V = 10A + 9, \quad W = 15A - 996,$$
$$F/25 = A^2 - 1182A + 39699 > 0 \text{ if } A \geqq 1148.$$

Conditions (7) hold if $A\geqq 76$. It was verified that $E(A)=0$ for $A<1148$.

Finally, let $m=6$. By (1),

$$1 + 3e(x - k) = (3x - 3k + 1)^2.$$

Hence the following equations are equivalent:

$$(11) \quad A = \sum_4 e(x - k), \quad 3A + 4 = \sum_4 (3x - 3k + 1)^2.$$

When the former holds we write $E_k(A)=0$, attaching the subscript k to the earlier E. Then also $E_g(A)=0$ if $g>k$. Hence $E_g(A)>0$ implies $E_k(A)>0$ for every $k\leqq g$. Let k be any given integer $\geqq 0$. To prove that $E>0$, it evidently suffices to exhibit one positive integer A for which $E_k(A)>0$. For a sufficiently large integer n,

$$g = \tfrac{1}{3}(5\cdot 4^n - 2) \tag{12}$$

is an integer $\geqq k$. Take $3A+4=4^{2n}\cdot 46$. Then $E_g(A)>0$ by the following lemma.

LEMMA 1. *There do not exist four integers $x\geqq 0$ satisfying*

$$4^{2n}\cdot 46 = \sum_4 (3x + 3 - 5\cdot 4^n)^2. \tag{13}$$

For $n=0$, $46=\sum(3x-2)^2$ is not solvable. In fact, the summands <46 are $(-2)^2$, 1^2, 4^2. If $2\cdot 23$ is a sum by four of 1, 4, 16, the value 1 must be used twice, while the maximum sum by two of 4 and 16 is $32<46-2$. For $n\geqq 1$, Lemma 1 follows by induction from $n-1$ to n. Let (13) hold. Since the sum of the four squares is a multiple of 8, each square is even. Thus x is odd, $x=2y+1$, where y is an integer $\geqq 0$. Cancellation of 4 gives

$$4^{2n-1}\cdot 46 = \sum(3y + 3 - 10\cdot 4^{n-1})^2.$$

The four squares are again all even, whence $y=2z+1$, where z is an integer $\geqq 0$. Cancellation of 4 now gives a result like (13) with n replaced by $n-1$. Hence the induction is complete.

By Theorem 3, $E_2(B)\leqq 1$ for every $B\geqq 0$. This completes the proof that $E=1$ for every $k\geqq 2$.

It is an interesting fact that g in (12) is the greatest integer for which $E_g(A)>0$ when $3A+4=4^{2n}\cdot 46$. This is a consequence of $E_{g+1}(A)=0$. The latter follows from

LEMMA 2. *There exist four integers $x\geqq 0$ satisfying*

$$4^{2n}\cdot 46 = \sum(3x - 5\cdot 4^n)^2. \tag{14}$$

This is true for $n=0$ since 46 is the sum of 25, 4, 1, 16, which are the squares of the values of $3x-5$ for $x=0, 1, 2, 3$. For $n\geqq 1$, we proceed by induction from $n-1$ to n. Hence assume (14) with n replaced by $n-1$. Multiply the assumed equation by 4^2 and write $\xi=4x$; we get (14) with x replaced by ξ.

Instead of Lemmas 1 and 2, we may employ

LEMMA 3. *If n is an odd integer $\geqq 3$ and $C=4^{n-3}\cdot 88$, then*

$$C \neq \sum(3x+3-2^n)^2, \quad C = \sum(3x-2^n)^2, \tag{15}$$

each summed for four integers $x \geqq 0$. In other words, if $t=(2^n-2)/3$ and $3A+4=C$, then $E_t(A)>0$, $E_{t+1}(A)=0$.

The proof is by induction from $n-2$ to n and is like that for Lemmas 1 and 2 except for the initial value $n=3$. For $n=3$, the squares in (15) must be even, whence $x=2y+1$ in (15_1) and $x=2y$ in (15_2), and the relations become

$$22 \neq \sum(3y-1)^2, \quad 22 = \sum(3y-4)^2.$$

These follow since 22 is not a sum by four of 1, 4, 25, $\cdots$, while $22=(-4)^2+2(-1)^2+2^2$.

For $n=1$, $g=6$; for $n=5$, $t=10$. Hence

$$E_i(244) > 0 \text{ if } i < 7, \quad E_7(244) = 0,$$
$$E_j(468) > 0 \text{ if } j < 11, \quad E_{11}(468) = 0.$$

When $k=3$ or 4, the only numbers $A \leqq 1000$ for which $E_k(A) \neq 0$ are 244, 468, 500, 676, 852, 980. This holds also for $k=5$ or 6 except for 676, which is the sum of $e(7)=161$, $e(11)=385$, and the double of $p(5)=65$. For $k=7$, 8, 9, 10, the A's are 468, 852, 980, since 500 is the sum of $e(5)=85$, $e(9)=261$, $p(3)=21$, $p(7)=133$. Next,

$$852 = 2e(5) + 2p(11), \quad 980 = e(5) + e(13) + p(3) + p(11),$$

since $p(11)=341$, $e(13)=533$. Hence $E_{11}(B)=0$ if $B \leqq 1000$.

7. THEOREM 5. *If $m=6$, $k=3$, $A \not\equiv 4 \pmod 8$, then $E(A)=0$.*

Here $A=3a-16b+84$. If A is odd, $d=6$, $P=28$,

$$U = 9\cdot 16(A-39), \quad V = 12A+16, \quad W = 18A-800,$$
$$F/36 = (7A-128)^2 - 4(A-39)V = (A+8)^2 + 64\cdot 294.$$

Conditions (7) hold if $A \geqq 88$. The numbers before $16m$ in Table IV include all $\leqq 88$ except

$$28 = 1 + 1 + 5 + 21, \quad 52 = 2\cdot 5 + 2\cdot 21,$$
$$60 = 1 + 5 + 21 + 33, \quad 84 = 4\cdot 21.$$

Next, let $A \equiv 2 \pmod 4$. Take $b = 2B$. Then a is an integer if $B \equiv A \pmod 3$, and $a \equiv 2 \pmod 4$. Then $B^2 < a$ if

$$3B < 16 + v^{1/2}, \qquad v = 3A + 4.$$

Lemma 3 in this Bulletin for January (vol. 34, pp. 63–72) applies if

$$4B > 15 + u^{1/2}, \quad u = 4A - 115 > 0, \quad A > 29.$$

The difference between the two limits for B exceeds 3 since $(A-77)^2 + 6936 > 0$.

Finally, let $A = 8p$. By §1, $24p+4$ is the sum of the squares of four integers $6x-5$ with $x \geqq 0$. Write $y = 2x+1$. Thus $3A+4$ is the sum of the squares of four integers $3y-8$ with $y \geqq 0$. By (11) with $k=3$, this implies $E(A) = 0$.

8. THEOREM 6*. *For every $k \geqq 0$, there exist integers $A > 0$ such that $E_k(A) > 0$ for the octagonal function $p_8(x-k)$.*

In (11) we replace $x-k$ by $h-x$ and conclude that

$$(16) \quad A = \sum_4 p(x-h), \quad 3A + 4 = \sum_4 (3x - 3h - 1)^2$$

are equivalent equations. Choose an integer $n \geqq 0$ such that $g \geqq \frac{1}{2}k$, for g defined by (12). Then $6q+4 = 4^{2n}\cdot 46$ determines an integer $q > 0$. Take $A = 8q+4$. We shall prove that $E_{2g}(A) > 0$, whence $E_k(A) > 0$. Suppose that $E_{2g}(A) = 0$, so that (16_2) holds for $h = 2g$. But $3A+4 = 24q+16$. Hence the four squares are all even and $x = 2y+1$. Cancellation of 4 gives

$$4^{2n}\cdot 46 = \sum (3y - 3g + 1)^2 = \sum (3y + 3 - 5\cdot 4^n)^2,$$

in contradiction with Lemma 1. We may vary this proof by taking $2g - 1 \geqq k$, or by taking $6q+4$ to be C of Lemma 3 and choosing n so that $2t \geqq k$.

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* This completes the proof of Theorem 5 of the author's paper in this Bulletin for January (vol. 34, pp. 63–72).