NOTE ON INTERCHANGE OF ORDER OF LIMITS*

BY T. H. HILDEBRANDT

There exist a number of theorems giving necessary and sufficient conditions for the relation

$$\lim_m \lim_n a_{mn} = \lim_n \lim_m a_{mn},$$

$\{a_{mn}\}$ being a double sequence of real numbers. Most of these theorems are not symmetric in m and n, which is only natural, because as a rule wherever the interchange of order of limits is in question, there is information about the iterated limit in one order, and the existence of the limit in reverse order is desired. Nevertheless it may be of interest to deduce a condition which is symmetric in m and n.

By way of notation, we shall assume that the symbol $\lim_m \lim_n a_{mn}$ implies that for every m, $\lim_n a_{mn}$ exists. On the other hand, we shall use the symbol $\lim_m \overline{\underline{\lim}}_n\, a_{mn}$ with the implication that

$$\lim_m \overline{\lim}_n\, a_{mn} = \lim_m \underline{\lim}_n\, a_{mn}.$$

Then, obviously, if we deduce necessary and sufficient conditions for the relation

$$\lim_m \overline{\underline{\lim}}_n\, a_{mn} = \lim_n \overline{\underline{\lim}}_m\, a_{mn},$$

those for the equality mentioned at the outset require in addition the assumption of the existence of $\lim_n a_{mn}$ for every m and of $\lim_m a_{mn}$ for every n.

Let us deduce necessary conditions. Let $\overline{\lim}_m\, a_{mn} = b_n$ and $\underline{\lim}_m\, a_{mn} = c_n$, and $\lim_n b_n = \lim_n c_n = d$. Then for every $\epsilon > 0$, there exists an n_ϵ such that when $n \geqq n_\epsilon$, we have

$$|b_n - d| \leqq \epsilon \text{ and } |c_n - d| \leqq \epsilon.$$

From the definition of b_n and c_n we have, for every n, and

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every $\epsilon > 0$, an $m_{\epsilon n}$ (depending on ϵ and n) such that when $m \geqq m_{\epsilon n}$, we have

$$c_n - \epsilon \leqq a_{mn} \leqq b_n + \epsilon.$$

By combining these two statements, we obtain the following. For every $\epsilon > 0$, there exists an n_ϵ such that for every $n_1 \geqq n_\epsilon$, there exists an $m_{\epsilon n_1}$ such that for every $m_2 \geqq m_{\epsilon n_1}$, we have

$$d - 2\epsilon \leqq a_{m_2 n_1} \leqq d + 2\epsilon,$$

i. e.,

$$\left| a_{m_2 n_1} - d \right| \leqq 2\epsilon.$$

For future reference we shall call this Statement A. Similarly, from the fact that $\lim_m \overline{\underline{\lim}}_n\, a_{mn} = d$, we have: For every $\epsilon > 0$ there exists an m_ϵ such that for every $m_1 \geqq m_\epsilon$, there exists an $n_{\epsilon m_1}$ such that for every $n_2 \geqq n_{\epsilon m_1}$, we have

$$\left| a_{m_1 n_2} - d \right| \leqq 2\epsilon.$$

By combining these two statements into one, and replacing 4ϵ by ϵ, we get the necessary condition desired, viz.: For every $\epsilon > 0$, there exists an n_ϵ and an m_ϵ such that for every $n_1 \geqq n_\epsilon$ and $m_1 \geqq m_\epsilon$, there exists an $m_{\epsilon n_1}$ and an $n_{\epsilon m_1}$ such that for every $m_2 \geqq m_{\epsilon n_1}$ and $n_2 \geqq n_{\epsilon m_1}$, we have

$$\left| a_{m_1 n_2} - a_{m_2 n_1} \right| \leqq \epsilon.$$

This condition is also sufficient. For suppose the condition satisfied. Then for a particular m_1 and n_2 chosen in accordance with the specifications, and for every $n_1 \geqq n_\epsilon$ and every $m_2 \geqq m_{\epsilon n_1}$, we have

$$a_{m_1 n_2} - \epsilon \leqq a_{m_2 n_1} \leqq a_{m_1 n_2} + \epsilon.$$

From this we conclude that for every $n_1 \geqq n_\epsilon$, the greatest and the least of the limits b_n and c_n satisfy the conditions

$$a_{m_1 n_2} - \epsilon \leqq c_{n_1} \leqq b_{n_1} \leqq a_{m_1 n_2} + \epsilon.$$

This has as consequence that if n_1 and n_0 are greater than or equal to n_ϵ, then

$$\left| b_{n_0} - b_{n_1} \right| \leqq 2\epsilon, \quad \left| c_{n_0} - c_{n_1} \right| \leqq 2\epsilon, \text{ and } \left| c_{n_1} - b_{n_1} \right| \leqq 2\epsilon.$$

Hence $\lim_n c_n$ and $\lim_n b_n$ exist and are equal, i.e. $\lim_n \overline{\underline{\lim}}_m a_{mn}$ exists. From the symmetry of the condition, we conclude that $\lim_m \overline{\underline{\lim}}_n a_{mn}$ exists also. The identity of the two limits is then a consequence of the condition of our theorem and Statement A.

We note finally that the Cauchy condition for convergence of the double limit, $\lim_{mn} a_{mn}$, is the special case of our condition in which $m_{\epsilon n_1}$ and $n_{\epsilon m_1}$ are independent of n_1 and m_1 respectively, and can therefore be taken as m_ϵ and n_ϵ, respectively.

UNIVERSITY OF MICHIGAN

ON BOUNDED REGULAR FRONTIERS IN THE PLANE*

BY W. A. WILSON

1. *Introduction.* The term *regular frontier* has been introduced by P. Urysohn† to designate a continuum which is the frontier of two or more components of its complement. Regular frontiers in the plane have been discussed by various authors. A. Rosenthal‡ has shown that a continuum which is the union of two bounded continua that are irreducible between the same pair of points and have no other common points is a regular frontier. R. L. Moore§ has given necessary and sufficient conditions that a bounded continuum be a regular frontier whose complement has exactly two components. C. Kuratowski‖ has given necessary conditions for a continuum to be a regular frontier which is the frontier of *every* component of its complement.

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† P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7, p. 98.

‡ A. Rosenthal, *Teilung der Ebene durch Irreduzible Kontinua*, Sitzungsberichte der Münchener Akademie, 1919.

§ R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6, pp. 203–213.

‖ C. Kuratowski, *Sur les coupures du plan*, Fundamenta Mathematicae, vol. 6, pp. 130–145.