GENERALIZATIONS OF THE THEOREM OF FERMAT AND CAUCHY ON POLYGONAL NUMBERS*

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1. *Introduction and Summary.* We seek the least l such that every integer $A \geqq 0$ is a sum of l values of

$$(1) \qquad p_{m+2}(x-k) = \tfrac{1}{2}(x-k)[m(x-k-1)+2]$$

for integers $x \geqq 0$. When $k=0$, (1) is a polygonal number of order $m+2$, and Fermat stated that $l=m+2$. This was first proved by Cauchy, who found that all but four of the $m+2$ polygonal numbers may be taken as 0 or 1. A simpler proof was given by the writer in this Bulletin (vol 33 (1927), pp. 719–726) that paper will be cited as I.

When all but four of the l values of (1) are 0 or 1, we shall prove here that

$l=m-1$ if $k=1$, $m \geqq 7$; $l=m-2$ if $k=2$, $m \geqq 8$, or if $k \geqq 3$, $m \geqq 7$; $l=6$ if $k=1$, $m=6$, or $k=2$, $m=7$; $l=5$ if $k=1$, $m=4$ or 5, or if $k \geqq 2$, $m=6$; $l=4$ if $k \geqq 1$, $m=3$, or if $k \geqq 2$, $m=4$ or 5.

When all but s of the l values of (1) are 0 or 1, where $5 \leqq s \leqq l$, we shall prove that, for every $k \geqq 1$,

$$l = m-2 \quad \text{if} \quad m > 7, \quad l = s \quad \text{if} \quad m \leqq 7.$$

But A is a sum of l values of (1) if and only if†

$$(2) \qquad 8mA + l(m-2)^2 = \sum_{l} (2mx+2-m-2mk)^2,$$

summed for l integers $x \geqq 0$. We saw that $l=4$ when $k=1$, $m=3$, and when $k=2$, $m=4$ or 5. Hence (2) shows that for every integer $A \geqq 0$,

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† This implies the like formula with k replaced by any larger integer $k+p$. The values $x \geqq p$ give the same squares as occur in (2) for $x \geqq 0$. Hence we use the least permissible k.

$$24A + 4 = \sum_4 (6x - 7)^2, \quad 40A + 36 = \sum_4 (10x - 23)^2,$$
(3)
$$8A + 4 = \sum_4 (4x - 9)^2.$$

By Theorem 6 and (2), we have

(4) $$3A + 4 = \sum_4 (3x - 16)^2 \quad \text{if } A \not\equiv 4 \pmod 8.$$

For $s=5$, $m \leqq 7$, we saw that $l=5$. Then (2) for $k=1$ is

(5) $$8mA + 5(m-2)^2 = \sum_5 (2mx + 2 - 3m)^2.$$

For $m=3$ this result is obtained by adding 1 to each member of (3_1) and is discarded. When m is 4 or 6, we cancel factors 4 or 16. We get

(6) $$8A + 5 = \sum_5 (4x - 5)^2, \quad 40A + 45 = \sum_5 (10x - 13)^2,$$

(7) $$3A + 5 = \sum_5 (3x - 4)^2, \quad 56A + 125 = \sum_5 (14x - 19)^2.$$

The results obtained when $s=4$, $l=5$ are consequences of these.

When $k=1$, $m \geqq 8$, Theorem 2 states that $l=m-2$ except for $A=54m+16$. This is a remarkable fact since the single exceptional value may be made as large as we please by taking m sufficiently large. Again, if $k=5$, $m=6$, we find that $l=4$ fails first for $A=980$; by increasing k, the first failure will occur for a value of A exceeding any assigned number.

In later papers I shall prove that the value of l obtained when $s=3$ is never less than the minimum l found here, and I shall give an improved method valid when $s>4$, but not when $s=4$.

2. *Formulas*. Give (1) the notation $\frac{1}{2}mx^2+\frac{1}{2}nx+c$. Thus

(8) $$n = 2 - m - 2mk, \quad c = \tfrac{1}{2}m(k^2 + k) - k.$$

For $k \geqq 1$ the reduced minor conditions in I now follow from

(9) $$A \geqq 4c + 4E, \quad A \geqq 4c + \tfrac{2}{3}m.$$

For, from the inequality (9_2) we deduce (11) of I, since the sum $(-n)(2m+n)/m$ of its last two terms is negative, because $-n$ is positive and $2m+n$ is negative. By (8_1), we may write (4) of I in the form

$$(10)\qquad A = mg + 4c + b + r,\quad g = \tfrac{1}{2}(a - b - 2kb).$$

3. *Excess E when* $k=1$. The values of (1) for $x \geqq 0$ are $m-1$ and all the polygonal numbers of order $m+2$.

TABLE II.

SUMS BY FOUR OF $m-1$ AND POLYGONAL NUMBERS

0–4, $m-1$, m, $m+1$–5, $2m-2$, $2m-1$, $2m$, $2m+1$–6, $3m-3$, $3m-2$, $3m$, $3m+1$, 3–7, $4m-4$, $4m-1$, $4m+2$–8, $5m+1$, 2, 4, 5, 7, 8, $6m$, $6m+3$–9, $7m+3$–9, $8m+2$–10, $9m+1$, 4, 7–10, $10m+5$–11, $11m+4$–9, 11, $12m+3$, 4, 6–12, $13m+2$, 5, 7–12, $14m+6$–12, $15m+6$–9, 11–13, $16m+5$–13, $17m+4$, 5, 7–13, $18m+3$, 6, 7, 9–14, $19m+8$, 9, 11–14, $20m+7$, 10–14, $21m+7$–13, 15, $22m+6$–15, $23m+5$, 6, 8, 9, 11–15, $24m+4$, 7, 10–16, $25m+9$–13, 15, 16, $26m+8$, 10, 11, 13–16, $27m+9$, 11–16, $28m+8$–17, $29m+7$–13, 15–17, $30m+6$, 7, 9, 10, 12–17, $31m+5$, 8, 11–17, $32m+10$–18, $33m+9$, 10, 12, 13, 15–18, $34m+12$–18, $35m+11$–17, $36m+9$–19, $37m+8$–13, 15–19, $38m+7$, 8, 10, 11, 13–15, 17–19, $39m+6$, 9, 12–17, 19, $40m+11$, 12, 14–20, $41m+10$, 13, 15–20, $42m+13$–20, $43m+12$–17, 19, 20, $44m+11$–20, $45m+10$–13, 15–21, $46m+9$–21, $47m+8$, 9, 11–17, 19–21, $48m+7$, 10, 12–21, $49m+12$, 13, 15–21, $50m+11$, 14–22, $51m+13$–17, 19–22, $52m+13$–22, $53m+12$, 13, 15–21, $54m+17$–19, 21, 22.

If $m \geqq 6$, $E(9m+6)=2$, since neither $9m+5$ nor $9m+6$ is equal to a number of Table II. If $m=5$, Table II lacks $17=2m+7=3m+2=4m-3$. If $m=4$, it lacks $26=4m+10 = 5m+6 = 6m+2 = 7m-2$. Also,* $E(54m+16) = m-5$. Hence E is not smaller than the value shown in

THEOREM 1. *For* $k=1$, $E=m-5$ *if* $m\geqq 7$, $E=2$ *if* $m=6$, $E=1$ *if* $m=4$ *or* 5, $E=0$ *if* $m=3$.

By (10) we require that the values of $b+r$ shall include a complete set of residues modulo m when r takes the values $0, 1, \cdots, E$, and b takes certain consecutive odd values. When $m=3$ or $m\geqq 5$, this will be true if b takes the values

* And independently of Table II, since the only partitions of 53 into 0, 1, 3, 6, 10, 15, 21, 28, 36, 45 are 1+1+6+45, 1+1+15+36, 1+6+10+36, 1+3+21+28, 0+10+15+28, 1+10+21+21.

β, $\beta+2$, $\beta+4$, whence $d=6$. But when $m=4$, we need only β, $\beta+2$, whence $d=4$. Conditions (9) hold if $A \geqq 8m$.

Let $m \geqq 7$. By (6)–(15) of I,

$$U = 24mA - 63m^2 + 12m + 36, \quad V = 2mA - m^2 + 6m + 4,$$

$$P = 7m + 2, \quad W = 3mA - 14m^2 - 2m + 4,$$

$$F = m^2A^2 - 74m^3A - 28m^2A + 193m^4 + 70m^3 - 68m^2 - 24m > 0.$$

The last evidently holds if

$$A \geqq 74m + 28. \tag{11}$$

Next, let $m=6$. Then

$$U = 36(4A - 60), \; V = 12A - 8, \; P = 44, \; W = 18A - 512,$$

$$F = 36[(7A - 88)^2 - (12A - 8)(4A - 60)] = 36(A^2 - 480A + 7264),$$

and $F>0$ if $A \geqq 465$ and hence if (11) holds.

If $m \geqq 6$, Theorem 1 now follows from Lemma 3 of I and the following lemma.

LEMMA 1. *If $m \geqq 5$ in Table* II, *$E(A) \leqq m-5$ when A is between any consecutive blocks, while $E(A) \leqq 2$ when A is in any block.*

For, the difference between consecutive entries in any block is always $\leqq 3$. If r is the term free of m in the leader $qm+r$ of any block, then $r+4$ is found to be the term free of m of a number of Table II within the preceding block. Hence $qm+r-1$ is the sum of $m-5$ and the number $(q-1)m+r+4$ in the table.

When $m=5$, the last two sentences hold after we suppress from Table II all entries down to and including the last entry which differs by 3 from the next entry of the block. The only exception is the leader $9m+7$, while $9m+6$ exceeds $8m+10$ by 1. Hence $E(A) \leqq 1$ if $A \leqq 54m+22$.

$$\text{For } m = 5, d = 6, n = -13, c = 4,$$

$$U = 120A - 1479, \; V = 10A - 1, \; P = 37, \; W = 15A - 356,$$

$$F = 25A^2 - 10150A + 126685 > 0 \text{ if } A > 393 = 74m + 23.$$

These facts with Lemma 3 of I prove Theorem 1 for $m=5$.

$$\text{For } m = 4, d = 4, n = -10, c = 3,$$

$$U = 4(24A - 231),\ V = 4(2A - 1),\ P = 14,\ W = 12A - 140,$$

$$F = 16[(7A - 37)^2 + (2A - 1)(24A - 231)]$$
$$= 16(A - 16)^2 + 16 \cdot 882 > 0.$$

Finally, let $m=3$. Then $d=6$, $n=-7$, $c=2$,

$$U = 9(8A - 55),\ V = 6A + 1,\ P = 23,\ W = 9A - 128,$$

$$F = 9[(7A - 42)^2 - (6A + 1)(8A - 55)]$$
$$= 9(A^2 - 266A + 1819) > 0,$$

if $A \geqq 259$. For $A < 259$, Theorem 1 was verified by Tables I and II.

4. THEOREM 2. *If* $k=1$, $m \geqq 8$, $E(A) \leqq m-6$ *except for* $A=54m+16$.

Within every block of Table II the difference of any two consecutive entries is $\leqq 3$. If f is the term free of m in the leader $qm+f$ of any block having $q \neq 16$, 46, 52, 54, then $f+5$ is that of a number t occurring explicitly in the table. Hence $qm+f-1$ is the sum of $m-6$ and $t=(q-1)m+f+5$. For $q=16$, 46, and 52, a like result holds if we replace $f+5$ by $f+6$ and hence replace $m-6$ by $m-7$. Hence the theorem is true for $A < 54m+16$. For $54m+17 \leqq A \leqq 199m+37$, it is true by Lemmas 3 and 4 of I.

For $A \geqq 199m+38$, $E=m-6$, we have $d=8$,

$$U = 24mA - 63m^2 + 12m + 36,\quad V = 2mA - m^2 + 8m + 4,$$

$$P = 11m + 2,\qquad W = 3mA - 23m^2 - 4m + 4,$$

$$F = m^2A^2 - 200m^3A - 48m^2A + 562m^4 - 84m^3$$
$$- 264m^2 - 48m.$$

Then $F > 0$, in fact for $A = 198m+44+t$, $t \geqq 0$.

5. *Excess* E_s *when* $k=1$.

THEOREM 3. *When* $k=1$, $E_s=0$ *if* $m\leqq 7$, $s\geqq 5$, *and if* $m>7$, $s\geqq m-2$, *while* $E_s=m-s-2$ *if* $m>7$, $5\leqq s\leqq m-2$.

If $m\leqq 5$, this follows from Theorem 1. If $m>7$, $E_5\leqq m-7$ by Theorem 2 and the fact that $54m+16$ is the sum of the polygonal number $3m+3$ and the entry $51m+13$ of Table II. Since the polygonal numbers >1 exceed $m-1$, the summands yielding $m-2$ are all 1, whence $E_s(m-2)=m-2-s$. Hence $E_5=m-7$. If $s>5$, regard $s-5$ of the values 1 as polygonal numbers; hence $E_s\leqq m-7-(s-5)$. This proves Theorem 3 except for $m=6$ and 7.

For $m=6$ or 7, we seek a constant C_m such that $E_4(A)\leqq 1$ for every $A\geqq C_m$. When r takes the values 0 and $E=1$, and b takes 3 or 4 consecutive odd values according as $m=6$ or $m=7$, the values of $b+r$ include a complete set of residues modulo m, whence $d=6$ or 8, respectively.

In the discussion in § 3 for $m\geqq 7$, we had $d=6$, $E=m-5$. Hence it is valid here for $m=6$, $d=6$, $E=1$, and shows that $E_4(A)\leqq 1$ if $A\geqq 74m+28$. Likewise, the work at the end of § 4 is valid here for $m=7$, $d=8$, $E=1$, and shows that $E_4(A)\leqq 1$ if $A\geqq 198m+44$.

It remains to treat the values of A below these two limits. From Table II we readily deduce a list of the sums by five of $m-1$ and polygonal numbers and then verify for $m=6$ and $m=7$ that the list includes all positive integers $\leqq 54m+25$. From thence to $74m+28$, $E_4(A)\leqq 1$ if $m=6$ or 7 by Lemma 3 of I. If $m=7$ and $74m+20\leqq A\leqq 199m+37$, $E_4(A)\leqq 1$ by Lemma 4 of I. This completes the proof of Theorem 3.

6. *Excess* E *when* $k=2$. The values of (1) for $k=2$, $x\geqq 0$ are $m-1$, $3m-2$, and all the polygonal numbers of order $m+2$. Evidently $E(m-2)=m-6$. For $m=7$ the sums by four of the values mentioned were tabulated to 395; the only consecutive integers missing from the list are 393–4; hence $E(394)=2$, $E(A)\leqq 1$ if $A<394$. For $m=6$, $E(60)=1$. Hence E is not smaller than the value shown in

THEOREM 4. *For* $k=2$, $E=m-6$ *if* $m\geqq 8$, $E=2$ *if* $m=7$, $E=1$ *if* $m=6$, $E=0$ *if* $m=3$, 4, *or* 5. *If* $m=7$, $E(A)\leqq 1$ *except for* $A=394$.

Conditions (9) are satisfied if $A\geqq 16m$. For $r=0, 1, \cdots,$ E, we require that $b+r$ shall include a complete set of residues modulo m. When $m\geqq 8$, this will be true if b takes the values β, $\beta+2$, $\beta+4$, $\beta+6$; when $m=5$, also $\beta+8$; when $m=6$ or 7, only the first three.

First, let $m\geqq 8$. Then $d=8$,

$$U = 24mA - 135m^2 + 36m + 36, \quad P = 9m + 2,$$
$$V = 2mA - m^2 + 8m + 4, \quad W = 3mA - 27m^2 + 4,$$
$$F = m^2A^2 - 112m^3A - 40m^2A + 706m^4 + 188m^3 - 152m^2 - 48m.$$

Evidently $F>0$ if $A\geqq 112m+40$. For smaller A's exceeding $54m+16$, Lemmas 3 and 4 of I show that $E(A)\leqq m-6$ if $m\geqq 7$.

LEMMA 2. *If* $m\geqq 7$ *in Table* II, $E(A)\leqq m-6$ *except for* $A=9m+6$, $20m+9$, *and* $54m+16$.

From each block we suppress all entries down to and including the last entry which differs by 3 from the next entry. Proceed as in § 4. We now have the further exceptions $q=9$ and 20. Also

$$9m + 3 = m - 6 + t_1, \quad 9m + 5 = 1 + t_2,$$
$$20m + 8 = m - 6 + t_3, \quad 54m + 15 = m - 6 + t_4,$$

where the t_i occur in the table.

For $m\geqq 8$, Theorem 4 now follows since

$$9m + 6 = 1 + 2(3m + 3) + 3m - 2 + 1, \quad 20m + 9 = m + 2 + 6m + 4 + 10m + 5 + 3m - 2,$$
$$54m + 16 = 2(3m + 3) + 45m + 10 + 3m - 2 + 2.$$

For $m=7$, $E=2$, $d=6$, $P=37$,

$$U = 168A - 6327, \quad V = 14A - 3, \quad W = 21A - 962,$$
$$F/7 = 7A^2 - 826A + 131149,$$

and $F>0$ for every $A>0$. But if we attempt to use $E=1$, we have $d=8$; the discussion for $m\geqq 8$ now applies also for $m=7$ except when $A=54m+16=394$. This proves both parts of Theorem 4 if $m=7$.

Next, let $m=6$. Then $E=1$, $d=6$, $P=32$,

$$U = 9(16A - 512), \quad V = 4(3A + 1), \quad W = 18A - 704,$$
$$F/36 = (7A - 116)^2-(3A + 1)(16A - 512) = (A - 52)^2 + 11264.$$

The minor conditions in I are all satisfied if $A\geqq 44$. For $m=6$, Table II includes all numbers less than 44 except $33=1+2(3m-2)$.

Let $m=5$. Then $E=0$, $d=10$, $P=67$,

$$U = 120A - 3159, \quad V = 10A + 9, \quad W = 15A - 956,$$
$$F/25 = (A - 703)^2 - 457878 > 0 \text{ if } A \geqq 1380.$$

It was verified that every integer $\leqq 1380$ is a sum by four of 4, 13, and polygonal numbers of order 7, whence $E=0$.

Let $m=4$. Then $n=-18$, $c=10$, $A=2a-9b+40$. Our general method requires that a and b be odd and applies only when A is odd. By (10) with $r=0$, $b\equiv A \pmod 4$. Hence $d=4$, $P=6$,

$$U = 4(24A - 495), \quad V = 4(2A + 1), \quad W = 4(3A - 63),$$
$$F/16 = (7A - 61)^2 - (2A + 1)(24A - 495) = A^2 + 112A + 4216,$$

whence $F>0$ for every $A\geqq 0$. The minor conditions in I are all satisfied if $A\geqq 28$. For $m=4$, Table II includes all integers $\leqq 28$ except $26=m+2+2(3m-2)$.

Next, let A be even. Since b must be even, our previous method is not applicable.

LEMMA 3.* *If $b=2B$, a is even, $a-B^2$ is a sum of three squares, and*

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* Legendre, *Théorie des Nombres*, ed. 3, II, No. 628, with omission of denominator 2 in No. 629.

$$(12) \qquad 4B^2 + 2B + 1 > 3a,$$

then $a = \sum \alpha^2$, $b = \sum \alpha$ *have solutions* $\alpha, \beta, \gamma, \delta$ *in integers* $\geqq 0$.

Write $A = 2S$. Then $a = S + 9B - 20$. Inserting this value of a into (12) and multiplying by 16, we see that (12) holds if

$$(13) \qquad 8B > 25 + u^{1/2}, \quad u = 48S - 351 \geqq 0.$$

Similarly, $B^2 < a$ if

$$(14) \qquad 2B < 9 + v^{1/2}, \quad v = 4S + 1.$$

Take $B \equiv 2 - S \pmod 4$. Then $a \equiv 2 \pmod 4$, $a - B^2 \equiv 1$ or $2 \pmod 4$, according as B is odd or even. Hence $a - B^2$ is a sum of three squares, and all conditions in Lemma 3 are satisfied.

To insure the choice of B as a prescribed residue modulo 4, we require that the difference between the limits for B, determined by (13) and (14), shall exceed 4. The resulting inequality reduces as usual to

$$S^2 - 340S + 2440 > 0, \quad S \geqq 333.$$

It remains only to verify Theorem 4 for $m = 4$ when A is even and < 666. This was done above when $A \leqq 28$. Table II includes all integers from 28 to 238 except 236; Table I includes all from 235 to 666.

7. *Excess when* $k \geqq 3$. Evidently $E(m-2) = m - 6$. If $m = 7$,

$$394 = 54m + 16 = 6m - 3 + 2(10m + 5) + 28m + 8 + 1.$$

Hence Theorem 4 implies the cases $m \geqq 7$ of

THEOREM 5. *For* $k \geqq 3$, $E = m - 6$ *if* $m \geqq 7$, $E = 1$ *if* $m = 6$.

That $E = 1$ if $m = 6$ follows from Theorem 4 and a result to be proved in a later paper on extended polygonal numbers. It was verified that 116 (or 980) is the least positive integer which is not a sum by four of the values of (1) for $x \geqq 0$, $k = 4$ (or $k = 5$). For $x \geqq 0$, the values > 1 of (1) are all $\leqq m - 1$. Hence the proof in § 5 shows that Theorem 3 holds also when $k > 1$.

8. THEOREM 6. *If* $m=6$, $k=5$, $E(A)=0$ *except for* $A\equiv 4$ (mod 8).

If A is odd, our general method applies with

$$d = 6,\ U = 144(A - 119),\ V = 12A + 16,\ P = -4.$$

Then (14) and hence (13) of I holds. Also, (9) hold if $A \leqq 344$, which is below the limit 980 for which $E=0$ (§ 7).

Next, let $A\equiv 2$ (mod 4). Take $b=2B$ and determine B modulo 3 to make a an integer. Then $a\equiv 2$ (mod 4). Apply Lemma 3. Then (12) holds and $B^2<a$ if

$$4B > 31 + u^{1/2},\quad 3B < 32 + v^{1/2},\quad u = 4A - 403,$$
$$v = 3A + 4.$$

The difference between these limits for B exceeds d if

$$(15) \qquad 4v^{1/2} - 3u^{1/2} > R = 12d - 35.$$

Here $d=3$ and (15) holds for every A such that $u>0$.

Finally, let $A\equiv 0$ (mod 8). Take an odd B such that a is an integer, whence $d=6$. Then $a\equiv 4$ (mod 8) and Lemma 3 applies. The square of (15) holds if

$$(A - 491)^2 > 65712,\quad A \geqq 748.$$

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