It was found that

$$7^{N-1} \equiv 618,117,398,624,349,204,361,513,620,865,505,749 \pmod{N}.$$

Hence N is composite. This number furnishes another example of the scarcity of primes of this form. The next such number which has any chance of primality consists of 47 of the digits 1.

The second number tested is $(10^{41}+1)/11$ or

$$N = 9,090,909,090,909,090,909,090,909,090,909,090,909,091.$$

In this case it was found that

$$3^{N-1} \equiv 763,287,007,500,473,474,161,903,784,495,157,879,509 \pmod{N}.$$

It follows, then, that N is also composite. This result represents the sixth attempt and failure to discover a larger prime than $2^{127}-1$ found by Lucas in 1877.

THE UNIVERSITY OF CALIFORNIA

ON THE APPROXIMATE REPRESENTATION OF ANALYTIC FUNCTIONS*

BY DUNHAM JACKSON

The purpose of this paper is to discuss the convergence of approximating polynomials determined by a least-square criterion, together with certain auxiliary conditions. Let $f(x)$ be a given function over the interval $a \leqq x \leqq b$. For each positive integral value of n, let p_n be a positive integer $\leqq n$. A polynomial of the nth degree may be required, for example, to coincide in value with $f(x)$ at p_n specified points of the interval; and among the infinitely many polynomials of the

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nth degree satisfying this requirement, an approximating polynomial may be determined by the condition that the integral of the square of the error shall be a minimum.* Or it may be required that the polynomial and its first p_n-1 derivatives shall agree in value with $f(x)$ and its first p_n-1 derivatives at a single specified point. Or, more generally, contact of more or less high order may be prescribed at each of a number of different points, in such a way that the total number of auxiliary conditions adds up to p_n.

In carrying through the proof of convergence, it will be supposed that $f(x)$ is analytic over a certain region of the complex plane containing the interval (a, b) of the real axis in its interior. Some of the preliminaries, however, can be dealt with just as easily under less restrictive hypotheses.

The existence of a minimum in each of the cases to be considered is an almost immediate corollary of well known existence proofs for the problem without auxiliary conditions. For the main point of the proof there is to show that if an upper bound is assigned for the integral of the square of the error (with polynomials of given degree) the coefficients must belong to a bounded region, which may be taken as closed,† and the imposition of the new conditions merely restricts the choice of the coefficients to a closed manifold in this region. And the uniqueness of the minimizing polynomial is proved by the usual argument that if two different polynomials give equally good approximations their average gives a better one.‡ So these questions may be dismissed without further consideration.

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* For the case of trigonometric approximation, a problem of this character with auxiliary conditions independent of n has been treated in a recent paper by the writer: *Note on a problem in approximation with auxiliary conditions*, this Bulletin, vol. 32 (1926), pp. 259–262. The method there was quite different, the highly specialized character of the auxiliary conditions being compensated by greater generality in the hypotheses on $f(x)$.

† See, for example, D. Jackson, *On functions of closest approximation*, Transactions of this Society, vol. 22 (1921), pp. 117–128; pp. 118–121.

‡ See Transactions, loc. cit., pp. 121–122.

To proceed with a closer characterization of the approximating polynomials to be studied, let $f(x)$ be a given real function, defined for $a \leqq x \leqq b$, and n a given positive integer. For simplicity of statement, let it be supposed that $f(x)$ has $n+1$ continuous derivatives throughout (a, b). Then, if $P(x)$ is any polynomial, and if $P(x_1)=f(x_1)$ for a value of x_1 in the interval, the difference $f(x)-P(x)$, expressed by means of Taylor's theorem with the remainder, is either of the form $(x-x_1)^\nu \psi(x)$, where $1 \leqq \nu \leqq n$ and $\psi(x)$ is continuous and different from zero for $x=x_1$, or else it is of the form $(x-x_1)^{n+1}\psi(x)$, where $\psi(x)$ is bounded. In the former case the root is of multiplicity ν, and in the latter case, while not necessarily of determinate multiplicity, it may be said to be of multiplicity $\geqq n+1$.

Let $x_1, x_2, \cdots, x_q$ be q distinct points of the interval (a, b), where q is a positive integer $\leqq n$. With each of these points x_k, let a positive integer ν_k be associated, so that $\nu_1+\nu_2+\cdots+\nu_q=p\leqq n$. (If in particular $q=n$, of course each ν_k must be equal to 1.) Let $P_n(x)$ be a polynomial of the nth degree with real coefficients such that, for each x_k,

$$
\begin{aligned}
(1) \qquad P_n(x_k) = f(x_k), \quad P_n'(x_k) &= f'(x_k), \cdots, \\
P_n^{(\nu-1)}(x_k) &= f^{(\nu-1)}(x_k) \qquad (\nu = \nu_k).
\end{aligned}
$$

There would be one and just one polynomial of degree $p-1$ satisfying these conditions,* and since $p-1<n$, they are satisfied by infinitely many polynomials of the nth degree. In particular, let $P_n(x)$ be determined among all such polynomials of the nth degree by the requirement that

$$\int_a^b [f(x) - P_n(x)]^2 dx$$

shall be a minimum. *Then* $f(x)-P_n(x)$ *must be orthogonal to every polynomial* $\pi_n(x)$ *of the nth degree which vanishes with its first* ν_k-1 *derivatives at each of the points* x_k. For if there were a polynomial $\pi_n(x)$ satisfying these conditions, and not ortho-

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* See, for example, Markoff, *Differenzenrechnung*, Leipzig, 1896, pp. 1–3.

gonal to $f(x)-P_n(x)$, the polynomial $P_n(x)+h\pi_n(x)$ would satisfy the auxiliary conditions imposed on $P_n(x)$, for arbitrary h; the integral

$$\int_a^b [f(x) - P_n(x) - h\pi_n(x)]^2\,dx,$$

which satisfies the conditions for differentiating with regard to h under the sign of integration, would have a derivative reducing for $h=0$ to

$$-2\int_a^b [f(x) - P_n(x)]\pi_n(x)\,dx \neq 0\,;$$

and the integral would not have a minimum for $h=0$, as it must under the hypothesis that $P_n(x)$ is the minimizing polynomial.

From this it follows further that $f(x)-P_n(x)$ *must have roots of aggregate multiplicity* $\geqq n+1$ *in* (a, b). For if this were not the case, it would be possible to construct a polynomial $Q_n(x)$, of the nth degree or lower, having roots at the same points, with exactly the same multiplicities. Since the roots of $f(x)-P_n(x)$ must include those prescribed by (1), each with at least the specified multiplicity, $Q_n(x)$ would belong to the class of polynomials denoted by $\pi_n(x)$ in the preceding paragraph. But the supposed $Q_n(x)$ is not orthogonal to $f(x)-P_n(x)$, because $[f(x)-P_n(x)]Q_n(x)$ is not identically zero, and never changes sign, having all its roots of even order. So denial of the assertion in italics results in a contradiction.

The property expressed in this assertion can be made the basis of a convergence proof. Let it be supposed from now on that $f(x)$ is analytic throughout the interior and on the boundary of a circle of the complex plane with center on the axis of reals at the middle point of the interval (a, b), and with radius $R>2r$, where $r=(b-a)/2$, the function $f(x)$ being real for real values of x. Let a polynomial $P_n(x)$ be defined as above for each positive integral value of n, the numbers $q=q_n$, x_k, and ν_k being variable with n in any way, subject to the relations of inequality that have been imposed on them. *Then* $P_n(x)$ *converges toward* $f(x)$ *throughout the interior and on the boundary of any circle of radius* $\rho < R-2r$, *concentric with the first.* The theorem is stated and proved in this form, in spite of the fact that the region of convergence thus obtained does not necessarily include the whole interval (a, b); it will be possible to choose ρ so as to include the entire interval, if $R>3r$.

Let $y_1, y_2, \cdots, y_{n+1}$ be the $n+1$ roots, or $n+1$ of the roots, of $f(x)-P_n(x)$, multiple roots being indicated by repetition; it will not be necessary to specify what roots are distinct, and what roots are coincident. It is a matter of elementary algebra to see that*

$$\begin{aligned} \frac{1}{t-x} &= \frac{1}{t-y_1} + \frac{x-y_1}{t-y_1}\cdot\frac{1}{t-x} \\ &= \frac{1}{t-y_1} + \frac{x-y_1}{t-y_1}\cdot\frac{1}{t-y_2} + \frac{x-y_1}{t-y_1}\cdot\frac{x-y_2}{t-y_2}\cdot\frac{1}{t-x}. \\ &= \cdots, \end{aligned}$$

or, with the notation

$$g_k(t) = (t-y_1)(t-y_2)\cdots(t-y_k),$$

$$\frac{1}{t-x} = \frac{1}{g_1(t)} + \frac{g_1(x)}{g_2(t)} + \cdots + \frac{g_n(x)}{g_{n+1}(t)} + \frac{g_{n+1}(x)}{g_{n+1}(t)}\cdot\frac{1}{t-x}.$$

For any value of x inside the circle of radius R, by Cauchy's formula,

$$f(x) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-x},$$

the integral being extended around the circumference of the circle. Hence

$$f(x) = c_0 + c_1g_1(x) + \cdots + c_ng_n(x) + R_n(x)g_{n+1}(x),$$

* For the entire convergence proof, see Runge, *Theorie und Praxis der Reihen*, Leipzig, 1904, pp. 126–142; Hermite, *Sur la formule d'interpolation de Lagrange*, Journal für Mathematik, vol. 84 (1878), pp. 70–79; pp. 70–72.

where the c's are constants, and

$$R_n(x) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{g_{n+1}(t)(t-x)},$$

a function analytic throughout the interior of the circle. The terms $\sum c_k g_k(x)$ constitute a polynomial of the nth degree. Since $R_n(x)\ g_{n+1}(x)$ vanishes at the points $y_1, \cdots, y_{n+1}$, the polynomial agrees in value with $f(x)$ at these points, the roots of the difference between $f(x)$ and the polynomial having (at least) the multiplicities indicated by the repetitions among the y's. But these conditions determine a polynomial of the nth degree uniquely; *the polynomial* $\sum c_k g_k(x)$ *must be identical with* $P_n(x)$. Consequently

$$f(x) = P_n(x) + \int_C \frac{(x-y_1)(x-y_2)\cdots(x-y_{n+1})}{(t-y_1)(t-y_2)\cdots(t-y_{n+1})}\cdot\frac{f(t)dt}{t-x}.$$

Now let it be understood that x is in the interior or on the boundary of the circle of radius ρ, where, as already specified, $\rho<R-2r$. Then $|x-y_k|\leqq\rho+r$, for all values of k. On the other hand, $|t-y_k|\geqq R-r$, for all values of k and for all points t on the path of integration; and $|t-x|\geqq R-\rho$. So, if M is the maximum of $|f(t)|$ on this path,

$$|f(x) - P_n(x)| \leqq \frac{2\pi RM}{R-\rho}\cdot\left(\frac{\rho+r}{R-r}\right)^{n+1}.$$

Since $(\rho+r)/(R-r)<1$, the right-hand member approaches zero as n becomes infinite, and $f(x)-P_n(x)$ converges uniformly toward zero for all the values of x in question.

The foregoing discussion can be supplemented in various ways. The case $q_n=n+1$, which falls just outside the hypotheses, being a straightforward problem of interpolation with no place for a least-square condition, is the one treated by Runge in the passage cited. The case $q_n\leqq n$, $p=p_n=n+1$, also leaves no room for a least-square condition, but comes within the scope of the convergence proof.* In this case,

* See Hermite, loc. cit.

as well as in the preceding, it makes no essential difference if the points x_k, instead of being on the axis of reals, are anywhere in the circle of radius r. At the other extreme, the problem for $q_n = 0$ is that of the Legendre series, and while the part of the discussion that relates to auxiliary conditions is irrelevant, the proof of convergence is still valid, as far as it goes; it shows, for example, that the Legendre series for an entire function converges everywhere.* In one of the most interesting special cases under the earlier hypotheses, that in which $q_n = 1$, $p_n = n$ (or, a little less narrowly, $p_n = n - l$, where l is $\geqq 0$ and independent of n), the point x_1 being taken for simplicity as the middle point of the interval, Dr. D. V. Widder, with whom the writer discussed the substance of the paper in its preliminary stages, noted at once that the convergence takes place throughout the interior of the large circle of radius R; and instead of the requirement that $R > 2r$ it is sufficient to assume that $R > r$. More generally, if l is independent of n, and if there are for each value of n at least $n - l$ auxiliary conditions attached to points distant by not more that r' from the middle point of (a, b), where $r' < r$, it is allowable in the convergence proof to let ρ be any number less than $R - 2r'$, while the assumption with regard to R is that R is greater than the larger of the numbers r, $2r'$. To generalize in another direction, it would be possible to replace the square of the error by the mth power of its absolute value, $m > 1$, and to admit a weight-function in the integral to be minimized. An exhaustive discussion of questions of this sort would be outside the scope of the present paper.

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* See Whittaker and Watson, *Modern Analysis*, third edition, Cambridge University Press, 1920, pp. 322–323.