A GENERALIZATION OF RECURRENTS

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1. *Introduction.* It is well known that if

$$\phi(x) = \sum_{r=0}^{\infty} \phi_r x^r, \qquad \psi(x) = \sum_{s=0}^{\infty} \psi_s x^s$$

are two singly infinite series, then the coefficients in the expansion of $\phi(x)/\psi(x)$, $\log \phi(x)$, $e^{\phi(x)}$ can all be expressed as determinants in the quantities ϕ_r, ψ_s. These expressions are called *recurrents* and have been used by several writers* to evaluate determinants involving the binomial coefficients, Bernoulli numbers, etc.

In the present paper, the analogous results are given for the quotient of two *doubly* infinite series, and the logarithm and exponential of a doubly infinite series. The extension to m-tuply infinite series is briefly sketched in §8.

It is believed the expressions obtained are new; there is no reference to any such work in the four volumes of Muir's *History*. We assume throughout that all the series involved are absolutely convergent, so that the derangements and multiplications employed are justified. As a matter of fact, we are dealing essentially with infinite sets of quantities A_{rs}, B_{rs}, C_{rs}, $\cdots$, $(r, s, = 0, 1, 2, \cdots)$; the "variables" which appear in the series are merely convenient carriers for their coefficients.

We shall use, wherever convenient, the convention employed by writers on relativity for summations, namely,

$$U_{rs}x^r y^s,$$

which is taken to mean

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} U_{rs}x^r y^s,$$

the summations being understood.

* Muir's *History*, vols. II, III, IV, Chapters on recurrents.

2. *Degree and Rank*. Given a doubly infinite series

$$U(x,y) = U_{rs}x^r y^s, \tag{1}$$

we shall invariably write U in the order

$$U_{00} + U_{10}x + U_{01}y + U_{20}x^2 + U_{11}xy + U_{02}y^2 + \cdots,$$

or as

$$U(x,y) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} U_{l-k,k}x^{l-k}y^k. \tag{2}$$

We define

$$l = (l - k) + k, \qquad u_{lk} = \frac{l(l+1) + 2(k+1)}{2},$$

as the *degree* and *rank* respectively of the coefficient $U_{l-k,k}$. Hence the degree of a coefficient is the degree of the term it multiplies. The rank of a coefficient is simply its place in the series (2). For since from (2) there are $l+1$ terms of degree l, the coefficient U_{l0} appears in the $[(1+2+3+\cdots+l)+1]$st place, that is,

$$u_{l0} = \frac{l(l+1)}{2} + 1.$$

The coefficient $U_{l-k,k}$ is k terms to the right of U_{l0}, so that its rank is

$$\frac{l(l+1)}{2} + 1 + k = \frac{l(l+1) + 2(k+1)}{2} = u_{lk}.$$

Thus for U_{rs}, the degree is $r+s$, and the rank is

$$u_{rs} = \frac{(r+s)(r+s+1) + 2(s+1)}{2}. \tag{3}$$

Moreover, it follows from the meaning of rank, that given any positive integer n, the equation

$$n = u_{rs} \tag{4}$$

determines a unique pair of non-negative integers r, s, and hence a unique coefficient U_{rs} in the series (2). Let k be any integer not greater than $r+s$. Then, by (3),

$$u_{r+s-k,k} = \frac{(r+s)(r+s+1)+2(k+1)}{2};$$

hence

$$u_{r+s-k,k} + s - k = \frac{(r+s)(r+s+1)+2(s+1)}{2} = u_{rs}.$$

In particular

$$u_{r+s,0} + s = u_{rs}, \qquad s \leqq r+s. \tag{5}$$

3. *Coefficients for a Product.* If

$$A(x,y) = A_{qr}x^q y^r,$$

$$B(x,y) = B_{st}x^s y^t,$$

then we know that

$$A(x,y)\cdot B(x,y) = C_{uv}x^u y^v,$$

where

$$C_{uv} = \sum_{\sigma=0}^{u}\sum_{\tau=0}^{v} A_{u-\sigma,v-\tau}B_{\sigma\tau}. \tag{6}$$

4. *Coefficients for a Quotient.* Consider now

$$P(x,y) = P_{uv}x^u y^v,$$

$$Q(x,y) = Q_{qr}x^q y^r,$$

and let

$$\frac{P(x,y)}{Q(x,y)} = Z(x,y) = Z_{st}x^s y^t, \tag{7}$$

where the coefficients Z_{st} are to be determined.

First, we can assume $Q_{00} \neq 0$. For, if $Q_{00}=Q_{10}=Q_{01}=\cdots=0$, $Q_{ij}\neq 0$, multiply both sides of (7) by $x^i y^j$, replacing $Q(x, y)/x^i y^j$ by $Q'(x, y)$ and $x^i y^j Z(x, y)$ by $Z'(x, y)$ with $Z'_{00}=Z'_{10}=Z'_{01}=\cdots=0$, $Z'_{ij}=Z_{00}$. We then have a new equality of the same form as (7) with $Q'_{00}=Q_{ij}\neq 0$. Thus $P(x, y)=Q(x, y)Z(x, y)$, or, by (6),

$$P_{uv} = \sum_{\sigma=0}^{u}\sum_{\tau=0}^{v} Q_{u-\sigma,v-\tau}Z_{\sigma\tau}, \qquad (u,v = 0,1,2,\cdots). \tag{8}$$

It may be noted in passing, that just as recurrents are related to difference equations with constant coefficients, so may (7), if $Q(x, y)$ be a polynomial, be looked upon as a linear partial difference equation with constant coefficients to determine Z_{uv}.

Let us introduce the symbol

$$(\lambda_{r-k,k}\ ;\ u - r + k,\ v - k),$$

defined by the relations

$$(9)\quad \begin{cases} \lambda_{r-k,k} = \dfrac{r(r+1) + 2(k+1)}{2}, \\ (\lambda_{r-k,k}\ ;\ u - r + k, v - k) = 0, \ \text{if}\ r > u + k,\ \text{or}\ k > v, \\ \qquad = Q_{u-r+k,v-k},\ \text{if}\ k \leqq v\ \text{and}\ r \leqq u + k. \end{cases}$$

Then (8) may be written

$$(10)\qquad \sum_{r=0}^{u+v} \sum_{k=0}^{r} (z_{r-k,k}\ ;\ u - r + k, v - k) Z_{r-k,k} = P_{uv},$$

$$(p_{uv} = 1, 2, 3, \cdots).$$

For $\lambda_{r-k,k} = z_{r-k,k}$, by definition of rank in (3). Moreover setting $r-k=\sigma$, $k=\tau$ in (10), we see that every term that occurs in (8) occurs in (10), and conversely.

Finally, by virtue of (9) we may replace (10) by

$$(11)\qquad \sum_{r=0}^{n} \sum_{k=0}^{r} (z_{r-k,k}\ ;\ u - r + k, v - k) Z_{r-k,k} = P_{uv},$$

$$(p_{uv} = 1, 2, 3, \cdots),$$

where n is any integer $\geqq u+v$. Take $n = z_{ij}$ and consider the set of $n = p_{ij}$ equations (11), in the n unknowns Z_{00}, Z_{10}, $Z_{01}, \cdots, Z_{ij}$,

$$(12)\quad \begin{cases} Q_{00} Z_{00} & = P_{00}, \\ Q_{01} Z_{00} + Q_{00} Z_{10} & = P_{10}, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ Q_{ij} Z_{00} + (2\ ;\ i-1, j+1) Z_{10} + \cdots + Q_{00} Z_{ij} & = P_{ij}. \end{cases}$$

Since $Q_{00}\neq 0$, we have, solving for Z_{ij} by determinants,

$$
(13)\qquad Q_{00}^{n} Z_{ij} = \begin{vmatrix}
Q_{00} & 0 & 0 & \cdots & 0 & P_{00} \\
Q_{10} & Q_{00} & 0 & \cdots & 0 & P_{10} \\
Q_{01} & 0 & Q_{00} & \cdots & 0 & P_{01} \\
Q_{20} & Q_{10} & 0 & \cdots & 0 & P_{20} \\
Q_{11} & Q_{01} & Q_{10} & \cdots & 0 & P_{11} \\
Q_{02} & 0 & Q_{01} & \cdots & 0 & P_{02} \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
Q_{ij}, & (2\,;\,i-1,j+1), & (3,i-2,j+2) & \cdot & \cdot\ \cdot & P_{ij}
\end{vmatrix},
$$

where (13) is constructed on the following scheme. The elements in the sth column $(s<n)$ consist of $s-1$ zeros, then the coefficients of $Q(x, y)$ of degree zero, one, two, three, $\cdots$, in their proper order, the groups of coefficients of the same degree being separated by r' zeros, where $s=z_{r'-k',k'}$ determines r'. In fact, we see from (11), that the elements in the sth column are given by the expression

$$
(14)\qquad (s\,;\,u-r'+k',v-k'),
$$

where r' and k' are determined from the equation

$$
z_{r'-k',k'} = s,
$$

in accordance with (4), and (u, v) has the successive values

$$
(15)\quad (0,0),(1,0),(0,1),(2,0),(1,1),\cdots,(u,v),\cdots,(i,j),
$$

$n=\lambda_{ij}$ in number, (u, v) appearing in the λ_{uv}th place in (15). Now the $\sigma+1$ terms (uv) of constant degree $u+v=\sigma$, $(0\leqq\sigma\leqq i+j)$, appear in the order

$$
(16)\qquad (\sigma,0),(\sigma-1,1),\cdots,(\sigma-k,k),\cdots,(0,\sigma).
$$

If n is replaced u by $\sigma-v$, (14) becomes

$$
(s\,;\,\sigma-r'+k'-v,\ v-k'),
$$

so that for $v=0, 1, 2, \cdots, \sigma$, we have the values of (14) for the sequence (16). But this expression vanishes, by (9), unless

(a) $$v-k'\geqq 0,$$

(b) $$\sigma-r'+k'-v\geqq 0.$$

Thus the first k' terms of (16) yield k' zeros. Replace v by $r+k'$, where to satisfy (a) and (b) $0\leqq r\leqq\sigma-r'\geqq 0$. We thus obtain from the next $\sigma-r'+1$ terms of (15),

$$(s\ ;\ \sigma-r',0),(s\ ;\ \sigma-r'-1,1),\ \cdots,(s\ ;\ 0,\sigma-r'),$$

or by (9),

$$Q_{\sigma-r',0},\quad Q_{\sigma-r'-1,1},\ \cdots,\quad Q_{0,\sigma-r'},$$

the coefficients of $Q(x, y)$ of degree $\sigma-r'$ in their proper order. The remaining terms of (16) produce $\sigma+1-k'-(\sigma-r'+1)=r'-k'$ zeros.

Since the sequence of degree $\sigma+1$ following (16) produces k' zeros followed by

$$Q_{\sigma+1-r',0},\quad Q_{\sigma-r',1},\ \cdots,\ Q_{0,\sigma+1-r'},$$

we see that the coefficients of degree $\sigma-r'=0, 1, 2, 3, \cdots$ are separated by r' zeros, as stated. Q_{00} appears when $\sigma-r'=0$. From (a) and (b),

$$v=k',\qquad u=\sigma-v=r'-k'.$$

Hence Q_{00} appears in the $\lambda_{r'-k',k'}$ or the sth place in the column; i. e., in (13), the elements Q_{00} lie along the main diagonal. The last column in (13) consists of the elements $P_{00}, P_{10}, P_{01}, \cdots, P_{ij}$ in order of rank.

5. *Final Coefficients in the sth Column.* There is some doubt about the last few elements in the sth column, but this is obviated as follows. Take $s=z_{i+j-\tau,\tau}$, $(0\leqq\tau\leqq j-1)$, i. e., consider the $(n-j)$th, $(n-j-1)$th, $\cdots$, $(n-1)$th columns of (13).

We have then $s=n-j+\tau$ so that the sth column contains $n-j+\tau$ zeros, Q_{00}, followed by $i+j$ zeros by our results in §4. But since there are only n elements in the column, Q_{00} is followed by $j-\tau-1$ zeros, since $j-\tau-1$ is always less than $i+j$. Hence we can reduce (13) to a determinant of the $(n-j)$th order multiplied by Q_{00}^{j} to some power, for the j columns just considered consist entirely of zeros save along the main diagonal where Q_{00} appears.

Now $n=z_{ij}$ and $z_{i+j,0}+j=n$, by (5). Hence, setting $n-j=\nu$, we have

$$n - j = z_{i+j,0} = \frac{(i+j)(i+j+1)}{2} + 1 = \nu.$$

The elements in the νth row are now

$$Q_{ij},(2\,;\,i-1,j+1),(3\,;\,i-2,j+2),\cdots,\\ (\nu-1\,;\,i-\nu+2,j+\nu-2),P_{ij}$$

so that the sth column terminates with

$$(s\,;\,i-s+1,j+s-1)$$

and in the $(\nu-1)$th row the elements are

$$(z_{r-k,k}\,;\,k-r,\,i+j-1-k) = \delta_{r+1,k+i}Q_{0,i+j-1-r},\\ (r,k = 0,1,2,\cdots,i+j-1),$$

where δ_{uv} is the Kronecker symbol.

Thus we have

$$Q_{0,i+j-1},0,\ Q_{0,i+j-2},0,0,\ Q_{0,i+j-3},0,0,0,\ \cdots,\ Q_{00},P_{0,i+j-1},$$

so that our evaluation of Z_{ij} gives us

$$(17)\quad Q_{00}^{\nu}Z_{ij} = \begin{vmatrix} Q_{00} & 0 & \cdots & P_{00} \\ Q_{10} & Q_{00} & \cdots & P_{01} \\ Q_{01} & 0 & \cdots & P_{10} \\ Q_{20} & Q_{10} & \cdots & P_{20} \\ Q_{11} & Q_{01} & \cdots & P_{11} \\ Q_{02} & 0 & \cdots & P_{02} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ Q_{1,i+j-2} & \cdot & \cdots & \cdot \\ Q_{0,i+j-1} & 0 & \cdots & \cdot \\ Q_{ij} & (2\,;\,i-1,j+1), & \cdots & P_{ij} \end{vmatrix},$$

where

$$\nu = \frac{(i+j)(i+j+1)}{2} + 1.$$

6. *Expression of the Z's as Recurrents.* There remains still one more simplification; the quantities $Z_{i+j,0}$, $Z_{0,i+j}$ can be expressed as recurrents. For we obtain from (7), §4, by the ordinary multiplication rule

$$(18) \qquad P_{uv} = \sum_{t=0}^{u} \sum_{s=0}^{v} Q_{ts} Z_{u-t,v-s}, \ (u,v = 0,1,2,3, \cdots).$$

This result may be written

$$(19) \qquad P_{uv} - R_{uv} = \sum_{t=0}^{u} Q_{t0} Z_{u-t,v}, \qquad (u = 0,1,2,3, \cdots),$$

where

$$(20) \qquad R_{uv} = \sum_{t=0}^{u} \sum_{s=1}^{v} Q_{ts} Z_{u-t,v-s},$$

so that

$$R_{u0} = 0,$$

$$R_{u1} = \sum_{t=0}^{u} Q_{t1} Z_{u-t,0},$$

$$R_{u2} = \sum_{t=0}^{u} Q_{t1} Z_{u-t,1} + \sum_{t=0}^{u} Q_{t2} Z_{u-t,0},$$

etc.

The formula (19) gives for $u=0, 1, 2, 3, \cdots, i+j$, the set of $i+j+1$ equations

$$\begin{aligned}
&Q_{00} Z_{0v} && = P_{0v} - R_{0v},\\
&Q_{10} Z_{0v} + Q_{00} Z_{1v} && = P_{1v} - R_{1v},\\
&Q_{20} Z_{0v} + Q_{10} Z_{1v} + Q_{00} Z_{2v} && = P_{2v} - R_{2v},\\
&\quad\vdots \qquad\quad \vdots \qquad\quad \vdots && \\
&Q_{i+j,0} Z_{0v} + Q_{i+j-1,0} Z_{1v} + Q_{i+j-2,0} Z_{2v} + \cdots && \\
&\qquad\qquad + Q_{00} Z_{i+j,v} && = P_{i+j,v} - R_{i+j,v},
\end{aligned}$$

so that

$$(21)\qquad Q_{00}^{i+j+1} Z_{i+j,v} = \begin{vmatrix} Q_{00} & 0 & \cdots & P_{0v} - R_{0v} \\ Q_{10} & Q_{00} & \cdots & P_{1v} - R_{1v} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ Q_{i+j,0} & & \cdots & P_{i+j,v} - R_{i+j,v} \end{vmatrix} .$$

In particular

$$(22)\qquad Q_{00}^{i+j+1} Z_{i+j,0} = \begin{vmatrix} Q_{00} & 0 & \cdots & P_{00} \\ Q_{10} & Q_{00} & \cdots & P_{10} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ Q_{i+j,0} & & \cdots & P_{i+j,0} \end{vmatrix} .$$

which gives the required expression for $Z_{i+j,0}$ as a recurrent. From symmetry the expression for $Z_{0,i+j}$ is derived from (22) by simply interchanging the subscripts of all the terms in (21).

We may observe that, having obtained the quantities Z_{u0} by (22), we know R_{u1}, so that we can calculate the quantities $Z_{i+j-1,1}$ by means of (21). Proceeding thus step by step, we can finally calculate Z_{ij}.

There are a number of relations among the determinants (17), (22). For example, suppose we interchange x and y in the equation (7),

$$\frac{P(x,y)}{Q(x,y)} = Z(x,y).$$

The effect is merely to interchange the subscripts of the coefficients throughout. Hence in (17), we can interchange the subscripts of Z_{ij}, the Q's and the P's and obtain an expression for Z_{ji}, ν, the order of the determinant, being unaffected by the process. Again, we may write (7) as

$$\frac{P(x,y)}{Z(x,y)} = Q(x,y),$$

so that we can interchange the roles of the Q's and Z's in (17).

7. *Final Expressions for the Z's.* The first eleven coefficients in the development of

$$\frac{P(x,y)}{Q(x,y)} = Z_{00} + Z_{10}x + Z_{01}y + \cdots + Z_{03}y^3 + \cdots$$

are

$$Z_{00} = Q_{00}^{-1}P_{00},$$

$$Z_{10} = Q_{00}^{-2}\begin{vmatrix} Q_{00} & P_{00} \\ Q_{10} & P_{10} \end{vmatrix}, \qquad Z_{10} = Q_{00}^{-2}\begin{vmatrix} Q_{00} & P_{00} \\ Q_{01} & P_{01} \end{vmatrix},$$

$$Z_{20} = Q_{00}^{-3}\begin{vmatrix} Q_{00} & 0 & P_{00} \\ Q_{10} & Q_{00} & P_{10} \\ Q_{20} & Q_{10} & P_{20} \end{vmatrix}, \quad Z_{02} = Q_{00}^{-3}\begin{vmatrix} Q_{00} & 0 & P_{00} \\ Q_{01} & Q_{00} & P_{01} \\ Q_{02} & Q_{01} & P_{02} \end{vmatrix},$$

$$Z_{11} = Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & P_{10} \\ Q_{01} & 0 & Q_{00} & P_{01} \\ Q_{11} & Q_{01} & Q_{10} & P_{11} \end{vmatrix},$$

$$= Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & P_{01} \\ Q_{10} & 0 & Q_{00} & P_{10} \\ Q_{11} & Q_{10} & Q_{01} & P_{11} \end{vmatrix},$$

$$Z_{30} = Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & P_{10} \\ Q_{20} & Q_{10} & Q_{00} & P_{20} \\ Q_{30} & Q_{20} & Q_{10} & P_{30} \end{vmatrix},$$

$$Z_{03} = Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & P_{01} \\ Q_{02} & Q_{01} & Q_{00} & P_{02} \\ Q_{03} & Q_{02} & Q_{01} & P_{03} \end{vmatrix},$$

$$Z_{21} = Q_{00}^{-7} \begin{vmatrix} Q_{00} & 0 & 0 & 0 & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & 0 & 0 & 0 & P_{10} \\ Q_{01} & 0 & Q_{00} & 0 & 0 & 0 & P_{01} \\ Q_{20} & Q_{10} & 0 & Q_{00} & 0 & 0 & P_{20} \\ Q_{11} & Q_{01} & Q_{10} & 0 & Q_{00} & 0 & P_{11} \\ Q_{02} & 0 & Q_{01} & 0 & 0 & Q_{00} & P_{02} \\ Q_{21} & Q_{11} & Q_{20} & Q_{01} & Q_{10} & 0 & P_{21} \end{vmatrix},$$

$$Z_{12} = Q_{00}^{-7} \begin{vmatrix} Q_{00} & 0 & 0 & 0 & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & 0 & 0 & 0 & P_{01} \\ Q_{10} & 0 & Q_{00} & 0 & 0 & 0 & P_{10} \\ Q_{02} & Q_{01} & 0 & Q_{00} & 0 & 0 & P_{02} \\ Q_{11} & Q_{10} & Q_{01} & 0 & Q_{00} & 0 & P_{11} \\ Q_{20} & 0 & Q_{10} & 0 & 0 & Q_{00} & P_{20} \\ Q_{12} & Q_{11} & Q_{02} & Q_{10} & Q_{01} & 0 & P_{12} \end{vmatrix}.$$

8. *Quotient of two m-tuply Infinite Series.* The same method can be applied to the development of the quotient of two triply, or indeed of two m-tuply infinite series. We need only to generalize the formulas for degree and rank of §2, for product of two series in §3 and to introduce a symbol corresponding to the $(\lambda_{r-k,k};\, u-r+k,\, v-k)$ of §4 to obtain the analog of (17); the analog of (21) is obtained with equal ease. Thus for the m-tuply infinite series,

$$A(x_1, \cdots, x_m) = A_{i_1 i_2 \cdots i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m},$$

$i_1+i_2+\cdots+i_m$ is the *degree* of the coefficient A_i above and

$$(23)\quad a_{i_1 i_2 \cdots i_m} = \sum_{r=1}^{m} \binom{i_r + i_{r+1} + \cdots + i_m + m - r}{m - r + 1} + 1$$

is its *rank*, when A is written so that the terms of degree $0, 1, 2, \cdots, r, r+1, \cdots$ succeed each other in order, and the terms of degree r are arranged in alphabetic order.

For the product of two such series, we have the formula

$$A(x_1, \cdots, x_m) \cdot B(x_1, \cdots, x_m) = C(x_1, \cdots, x_m),$$

where

$$(24) \qquad C_{j_1, \cdots, j_m} = \sum_{r=0}^{j} A_{j_1-r_1, j_2-r_2, \cdots, j_m-r_m} B_{r_1, r_2, \cdots, r_m}.$$

If we define $Z(x_1, \cdots, x_m)$ by

$$(25) \qquad \frac{P(x_1, \cdots, x_m)}{Q(x_1, \cdots, x_m)} = Z(x_1, \cdots, x_m),$$

then our new symbol is

$$\Delta_{sj} = (\lambda_{sj}; j_1 - s_1 + s_2, j_2 - s_2 + s_3, \cdots, j_{m-1} - s_{m-1} + s_m, j_m - s_m),$$

defined by $\Delta_{sj} = 0$ if $j_r - s_r + s_{r+1}$ is negative for any r between 0 and $m+1$, and

$$(26) \qquad \Delta_{sj} = Q_{j_1-s_1+s_2, \cdots, j_m-s_m},$$

if $j_r - s_r + s_{r+1}$ is positive for every r between 0 and $m+1$, and by convention $s_{m+1} = 0$. But our final results in the general case are completely obscured by the symbolism introduced to express them.

9. *Expansion of a Logarithm.* We can readily obtain the expansion of

$$(27) \qquad \log Q(x, y) = Z(x, y)$$

where

$$Q(x, y) = Q_{qr} x^q y^r, \qquad Q_{00} \neq 0,$$
$$Z(x, y) = Z_{st} x^s y^t.$$

For, operating on (27) with

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

we obtain a result of the form

$$\frac{P(x,y)}{Q(x,y)} = W(x,y),$$

where

$$W_{st} = (s + t)Z_{st}, \qquad P_{uv} = (u + v)Q_{uv},$$

by Euler's theorem on homogeneous functions and our convention as to the order of an infinite series.

Thus $Z_{00} = \log Q_{00}$; the other coefficients are derived from our previous expressions by replacing Z_{ij} by $Z_{ij}/(i+j)$ and P_{uv} by $(u+v)Q_{uv}$.

10. *Expansion of an Exponential.* For $e^{Q(x,y)}$, a slightly different procedure is necessary. Let

$$(28) \qquad \begin{cases} e^{Q(x,y)} = W(x,y), \\ \theta = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} \cdot \end{cases}$$

We shall have

$$(29) \qquad \begin{cases} Q(x,y) = Q_{qr}x^q y^r, & \theta Q = (q + r)Q_{qr}x^q y^r, \\ W(x,y) = W_{st}x^s y^t, & \theta W = (s + t)W_{st}x^s y^t. \end{cases}$$

Then

$$(30) \qquad \theta e^Q = \theta Q e^Q = (\theta Q) \cdot W = \theta W.$$

Hence, by (6),

$$(31) \quad (u + v)W_{uv} = \sum_{\sigma=0}^{u} \sum_{\tau=0}^{v} (u + v - \sigma - \tau)Q_{u-\sigma,v-\tau}W_{\sigma\tau}.$$

Now as in §4 introduce a symbol

$$(\lambda_{r-k,k} \,;\, u - r + k, v - k)$$

defined as in (9), with one modification, namely, while

$$\lambda_{r-k,k} = \frac{r(r + 1) + 2(k + 1)}{2}$$

and

$$(\lambda_{r-k,k}\,;\,u-r+k,v-k)=0,\ \text{ if }\ r>u+k,\ \text{ or }\ k>v,$$
$$=(u+v-r)Q_{u-r+k,v-k},$$
$$\text{if }\ k\leqq v \text{ and } r\leqq u+k,$$

we have

$$(\lambda_{r-k,k}\,;\,u-r+k,v-k)=-(u+v),$$

for $k=v$ and $r=u+v$. Then (31) may be written in the form

$$\sum_{r=0}^{u+v}\sum_{k=0}^{r}(W_{r-k,k}\,;\,u-r+k,v-k)W_{r-k,k}=0, \tag{32}$$

just as (10) was equivalent to (8) in §4. Also $(\lambda_{00};0,0)=0$, but from (28) we see that

$$e^{Q_{00}}=W_{00}=-P_{00},\ \text{say}.$$

Thus (28) becomes equivalent to (10) if we replace in each equation P_{uv} by 0 for $u+v>0$ and $(\lambda_{uv};0,0)$ by $-(u+v)$, instead of by Q_{00}. We thus obtain the following set of w_{ij} equations for W_{ij}:

$$\begin{aligned}
&-W_{00} &&=P_{00},\\
&1\cdot Q_{10}W_{00}-1\cdot W_{10} &&=0,\\
&1\cdot Q_{01}W_{00}+0\cdot W_{10}-1\cdot W_{01} &&=0,\\
&2\cdot Q_{20}W_{00}+1\cdot Q_{10}W_{10}+0\cdot W_{01}-2W_{20} &&=0,\\
&2\cdot Q_{11}W_{00}+1\cdot Q_{01}W_{10}+1\cdot Q_{10}W_{01}+0\cdot W_{20}-2W_{11} &&=0,\\
&2\cdot Q_{02}W_{00}+0\cdot W_{10}+1\cdot Q_{01}W_{01}+0\cdot W_{20} &&\\
&\qquad\qquad+0\cdot W_{11}-2W_{02} &&=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &&\\
&(i+j)Q_{ij}W_{00}+\cdots \qquad -(i+j)W_{ij} &&=0.
\end{aligned}$$

The determinant of this system is

$$(-1)(-1)^2(-2)^3(-3)^4\cdots(-i-j+1)^{i+j}(-i-j)$$
$$=(-1)^{w_{ij}}1^2\cdot 2^3\cdot 3^4\cdots(i+j-1)^{i+j}(i+j)^i.$$

Just as before, if we solve for W_{ij}, we can reduce the determinant we obtain corresponding to (12) to one of the νth order,

$$\nu=\frac{(i+j)(i+j+1)}{2}+1.$$

But we can also develop this expression with respect to its last row which is P_{00}, 0, 0, $\cdots$, 0, obtaining a determinant of the $(\nu-1)$st order with a factor $(-1)^{\nu-1}$. Hence our final form for W_{ij} is

$$1^2\cdot 2^3\cdot 3^4\cdots(i+j-1)^{i+j}W_{ij}$$
$$=-\begin{vmatrix} Q_{10} & -1 & 0 & 0 & 0 & \cdots & 0\\ Q_{01} & 0 & -1 & 0 & 0 & \cdots & 0\\ 2Q_{20} & Q_{10} & 0 & -2 & 0 & \cdots & 0\\ 2Q_{11} & Q_{01} & Q_{10} & 0 & -2 & \cdots & 0\\ 2Q_{02} & 0 & Q_{01} & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ & & & & & & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & & -(i+j-1)\\ (i+j)\,Q_{ij} & (2,i-1,j+1) & & & & \cdots & \end{vmatrix}.$$

As before, we can interchange subscripts of all the Q's to obtain W_{ji}. We can also express $W_{i+j,0}$ and $W_{0,i+j}$ as recurrents; thus

$$(i+j-1)!W_{i+j,0} = -\begin{vmatrix} Q_{10} & -1 & & & & 0 \\ 2Q_{20} & Q_{10} & -2 & & & 0 \\ 3Q_{30} & 2Q_{20} & Q_{10} & -3 & & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ & & & & & -(i+j) \\ (i+j)Q_{i+j,0} & \cdots & & & & Q_{10} \end{vmatrix}$$

with a similar expression for $W_{0,i+j}$.

11. *Conclusion.* It hardly seems necessary to give numerical examples of these expansions. As in the case of recurrents, from expressions of such generality any desired example may be derived by a mere substitution of numbers for letters in the general formulas. The quotient of two polynomials, the reciprocal of a series or a polynomial, for example, are included as special cases.

It appears from the expressions for Z_{21} and Z_{12} in §7, that a further immediate reduction of the order of the determinants (17) is sometimes possible; but to explicate this reduction in the general case would be to mar the simplicity and symmetry of our developments.

In conclusion I should like to thank Professor E. T. Bell for criticism and suggestions in the writing of this paper.

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A CORRECTION

In the paper by H. W. March, *The Heaviside operational calculus*, this Bulletin, vol. 33(1927), on page 312, in the line following equation (2), change "*negative*" to "*positive.*"