generated by any finite number of H's. Let $t'(\neq 1)$ be an element common to K and any other H_t, so that we have $t' = s_1' s_2' \cdots s_k'$, where s_i' is in H_{s_i}. Let t be any other element of H_t. Then, by Theorem 5, there exists an element u of H_t and integers m and n such that $t = u^m$ and $t' = u^n$. Under the condition imposed on G there exist in G elements v_i, such that $v_i^n = s_i'$, for every i from 1 to k, and each of these v_i are, of course, in H_{s_i}. We have, then,

$$t' = s_1' s_2' \cdots s_k' = (v_1 v_2 \cdots v_k)^n = u^n.$$

But in a group G containing no elements of finite order the last equality implies $u = v_1 v_2 \cdots v_k$. Hence, u is in K and, hence, t is in K. We have, therefore

THEOREM 10. *The necessary and sufficient condition that a proper primitive partition of an abelian group G containing no elements of finite order be an L-partition is that every element of G have a kth root in G for every integer k.*

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ANALYTIC FUNCTIONS WITH ASSIGNED VALUES*

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1. *Introduction.* The question of determining when the values of a function at a denumerably infinite set of points in a finite region determine a function analytic in this region, and if so the power series for the function in question, has recently been raised by Professor T. H. Hildebrandt.† It is well known that if the function in question exists, it is uniquely determined.‡ The usual proof gives a process for determining the coefficients in the power series, in which

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* Presented to the Society, February 26, 1927.

† This Bulletin, vol. 32 (1926), p. 552.

‡ W. F. Osgood, *Funktionentheorie*, vol. 1 (1912), p. 337.

the expression for the nth coefficient involves n successive limits. We shall set up expressions involving single limits which determine the coefficients, and hence give a simpler process for obtaining the power series.

If the sequence be formed whose nth term is the nth root of the expression giving the nth coefficient, the existence and boundedness of the limits appearing in this sequence constitute a necessary and sufficient condition that the process leads to an analytic element. To insure that this element leads to a function of the kind sought, certain obvious conditions must be added.

2. *Necessary Conditions.* Consider a denumerably infinite set of points in a closed two-dimensional* region of the z plane. Let a function, $w=f(z)$, which is analytic in this region, be given. If its values at the points of the set are determined, we wish to see how the function may be recovered from these values. Since the given point set is infinite, and in a closed region, it has at least one limit point in the region. For simplicity, take one such limit point as the origin, designate some sequence of points in the set approaching the origin as a limit by $Z_1, Z_2, \cdots$, and the corresponding values of the function $f(z)$ by $W_1, W_2, \cdots$. These values alone uniquely determine the function.†

In a neighborhood of the origin, we have

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots. \tag{1}$$

Consequently, we may write

$$c_0 = \lim_{z\to 0} f(z), \quad \cdots \quad c_n = \lim_{z\to 0} \frac{f(z) - \sum_{k=0}^{n-1} c_k z^k}{z^n}. \tag{2}$$

* This is not an essential requirement, since a function analytic in all the points of a one-dimensional region is necessarily analytic in some closed two-dimensional region including this region.

† Osgood, loc. cit.

If we let z approach zero through the sequence Z_i, we have

$$(3) \qquad c_n = \lim_{i\to\infty} \frac{W_i - \sum_{k=0}^{n-1} c_k Z_i^k}{Z_i^n}.$$

While these formulas determine the coefficients successively, if we write them directly in terms of the given quantities, we see that the formula for the nth derivative involves n repeated limits. To find simpler formulas, involving single limits, we consider the expression

$$(4) \qquad A_n = \frac{\begin{vmatrix} f(z_1) & z_1^{n-1} & \cdots & z_1 & 1 \\ f(z_2) & z_2^{n-1} & \cdots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ f(z_{n+1}) & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}{\begin{vmatrix} z_1^n & z_1^{n-1} & \cdots & z_1 & 1 \\ z_2^n & z_2^{n-1} & \cdots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ z_{n+1}^n & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}.$$

This is suggested by the fact* that, for certain real functions $f(z)$, when the z_i in this expression close down on a point z_0, the limiting value of this expression is $f^{(n)}(z_0)/n!$. As the method of proof for the real case is based on Rolle's theorem, which has no simple analog for functions of a complex variable, we must here proceed otherwise.

Let us replace the functions $f(z_i)$, which occur in the numerator of (4), by their series expansions, as given by (1). In any circle $0 \leqq |z| \leqq R$ *inside* the circle of convergence of (1), these series converge uniformly, so that for values of z_i in this circle we may write

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* See the author's paper, *Osculating curves and surfaces*, Transactions of this Society, vol. 28 (1926), p. 402.

$$
(5) \qquad A_n = \sum_{k=0}^{\infty} c_k \frac{\begin{vmatrix} z_1^k & z_1^{n-1} & \cdots & z_1 & 1 \\ z_2^k & z_2^{n-1} & \cdots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ z_{n+1}^k & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}{\begin{vmatrix} z_1^n & z_1^{n-1} & \cdots & z_1 & 1 \\ z_2^n & z_2^{n-1} & \cdots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ z_{n+1}^n & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}.
$$

Of the determinants in the numerator, those in the terms for which $k \leqq n-1$ vanish. For $k \geqq n$, the numerator is divisible by the product of all the differences of the z_i, i. e., the Vandermonde determinant in the denominator. The quotient, S_{k-n}, is a homogeneous symmetric function of the z_i of the $(k-n)$th degree, and is readily found to be that in which all terms of this degree appear, each with coefficient unity. Consequently, if the z_i are in the circle $0 \leqq z \leqq r < R/n$, we shall have

$$
(6) \qquad S_{k-n} \leqq (|z_1| + |z_2| + \cdots + |z_n|)^{k-n} \leqq (nr)^{k-n}.
$$

Thus the series for A_n given in (5) is equivalent to

$$
(7) \qquad A_n = c_n + \sum_{k=n+1}^{\infty} c_k S_{k-n}.
$$

This shows that the series for $|A_n - c_n|$ is dominated by

$$
(8) \qquad \sum_{k=n+1}^{\infty} |c_k|(nr)^{k-n} = nr \sum_{k=n+1}^{\infty} |c_k|(nr)^{k-n-1}.
$$

But, for z in the circle $0 \leqq z \leqq R$, the series (1) as well as all the series obtained by termwise differentiation converge absolutely and uniformly.* In particular, from the $(n+1)$th derived series, for $z = R$, we find

$$
(9) \qquad \sum_{k=n+1}^{\infty} k(k-1) \cdots (k-n) |c_k| R^{k-n-1} = M.
$$

* Osgood, loc. cit., pp. 97–98, and p. 335.

Comparing this with (8), we see that

$$|A_n - c_n| \leqq nr\,M. \tag{10}$$

This shows that when r approaches zero, A_n approaches c_n.

Evidently r will approach zero provided the z_i close down to the origin. We note that if the limit point were z_0, not the origin, and we replaced z_i by $(\bar{z}_i - z_0)$ and $f(z_i)$ by $f(\bar{z}_i)$ in (4), this expression would remain of the same form as at present, with bars on the z's. Since $c_k = f^{(k)}(0)/k!$, we may state the following theorem.

THEOREM I. *If $f(z)$ is an analytic function, and $z_1, z_2, \cdots, z_n$ are n distinct points closing down on z_0, we shall have*

$$\lim A_n = f^{(n)}(z_0)/n!,$$

where A_n is given by (4) *or*

$$\sum_{j=1}^{n} \frac{f(z_j)}{(z_j - z_1)\cdots(z_j - z_{j-1})(z_j - z_{j+1})\cdots(z_j - z_n)}. \tag{11}$$

3. *Sufficient Conditions.* Theorem I suggests the following process for constructing the analytic function, if such exist, determined by the values on a denumerable point set. Select a sequence of points having a single limit point, take n consecutive points of this sequence as the z_i of Theorem I, and form the A_n, and their limits, c_n. This gives the series (1), which determines an analytic element provided its radius of convergence is not zero, i. e., that the maximum limit of $|c_n|^{1/n}$ is finite.* This gives the following theorem.

THEOREM II. *If the values of a function are given at a sequence of points having a single limiting point, and the A_n of Theorem* I *be formed from n consecutive points of the sequence, the existence of the limits*

$$\lim A_n^{1/n},$$

* E. J. Townsend, *Functions of a Complex Variable*, 1915, p. 232.

and the boundedness of these limits as a set, constitute a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} (\lim A_n) z^n$$

yield an analytic element.

COROLLARY. *If the values are given at any point set in a closed region, the conditions that they determine a function analytic in this region are* (1) *that for some subsequence the conditions of Theorem* II *are met,* (2) *that the resulting element is capable of analytic extension over the entire region, and* (3) *that the resulting function takes the correct values at all the given points.*

The third condition here seems stringent, but when it is recalled that if the value of the function at a single point of the region is changed, the required analytic function will no longer exist, it becomes evident that no essentially milder conditions will suffice.

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