$$C_{ij} = \frac{N_{ij}\left\{ \sum_{k=1}^{n} U_{xk}^{(i)} N_{ik} \right\}^{(n-2)/2}}{D^{(n-1)/2}}. \tag{18}$$

Now by a well known property of Jacobians,*

$$\sum_{j=1}^{n} \frac{\partial C_{ij}}{\partial x_j} = 0. \tag{19}$$

Hence, if in (19) the expressions on the right of (18) be substituted for C_{ij}, we will have the differential equation satisfied by $U^{(i)}$ alone. It is readily seen that the form of this equation is independent of the index (i) and hence the n functions

$$U^{(1)}, U^{(2)}, \cdots, U^{(n)}$$

satisfy the same differential equation, which may be looked upon as a generalization of Laplace's equation to curved n-space.

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THE NON-EXISTENCE OF A CERTAIN TYPE OF REGULAR POINT SET†

BY R. L. WILDER

In a paper not yet published,‡ I have shown that a regular§ connected point set which consists of more than one point and remains connected upon the omission of any connected subset, is a simple closed (Jordan) curve. As a simple closed curve is a bounded point set, it is clear that there does not exist any unbounded regular connected point set which remains connected upon the omission of any connected subset.

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* Muir, *Theory of Determinants*, vol. 2, p. 230.

† Presented to the Society, December 29, 1926.

‡ See, however, this Bulletin, vol. 32 (1926), p. 591, paper No. 35.

§ That is, connected im kleinen.

In the present paper I propose to consider the following question: *Does there exist any unbounded regular connected point set which remains connected upon the omission of any bounded connected subset?* If the additional restriction that it be closed is imposed upon the point set, J. R. Kline has shown* that the answer to this question is negative. In his proof Kline is able to make use of known properties of continuous curves. For the general case of non-closed sets these properties are not available, but by establishing certain properties of connected and regular point sets it is possible, as shown below, to give a negative answer to the above question.

DEFINITION. If M is a regular point set, a *region of* M is defined as follows: P being any point of M, and C a circle with center at P, then the set of all points of M which lie, with P, in a connected subset of M which lies within C is a *region of* M. It is clear that the set of all points of M which lie in a certain neighborhood of P are in a *region of* M.

DEFINITION. If A and B are two distinct points of a regular set M, *a simple chain of regions of* M *from* A *to* B is a finite sequence of regions of M, $R_1, R_2, \cdots, R_n$, such that (1) R_1 and R_2 contain A and B, respectively, (2) R_i $(i \neq 1, n)$ has points in common with R_{i-1} and R_{i+1}, but not with any other region of the sequence, (3) R_1 and R_n have points in common with R_2 and R_{n-1}, respectively, but not with any other region of the sequence.

THEOREM 1. *If* A *and* B *are any two distinct points of a regular connected point set* M, *then there exists a simple chain of regions of* M *from* A *to* B.

The proof of Theorem 1 is similar to the proof of Theorem 10 of R. L. Moore's *On the foundations of plane analysis situs.*†

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* *Closed connected sets which remain connected upon the removal of certain connected subsets,* Fundamenta Mathematicae, vol. 5 (1924), pp. 3–10.

† Transactions of this Society, vol. 17 (1916), pp. 131–164.

THEOREM 2. *If M is a regular point set, then any region of M is a regular point set.*

THEOREM 3. *If M is a regular connected point set, bounded or unbounded, and A and B are any two distinct points of M, then both A and B lie in a bounded, regular, connected subset of M.*

Theorem 3 is a consequence of Theorems 1 and 2.

THEOREM 4. *Let C_1 and C_2 be two mutually exclusive point sets, and M a regular connected point set which has at least one point in common with each of the sets C_1 and C_2, and such that the set of points common to M and C_i $(i=1, 2)$ is closed in M. Then there exists a point set K, subset of M, such that K is connected and bounded and contains no point of either C_1 or C_2, but such that C_1 and C_2 each contain at least one point of M which is a limit point of K.*

Theorem 4 is a generalization of the result contained in my paper *A theorem on connected point sets which are connected im kleinen.** Its proof, after an application of Theorem 3, is very similar to the proof of the result of the latter paper.

THEOREM 5. *If P is a point of a connected and regular point set M such that $M-P$ is the sum of two mutually separated† sets M_1 and M_2, then M_1+P and M_2+P are connected and regular sets.*

PROOF. Let K be any circle with center at P. Since M is regular, there exists a circle T concentric with K, such that all points of M interior to T lie with P in a connected subset of M which lies wholly interior to K. Denote by k the set of all points of M that lie with P in a connected subset of M which lies wholly interior to K, and by t the set of all points of M that lie interior to T. Clearly t is a subset of k.

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* This Bulletin, vol. 32 (1926), pp. 338–340.

† Two sets are said to be mutually separated if they are mutually exclusive and neither contains a limit point of the other.

By a theorem due to Knaster and Kuratowski,* M_1+P and M_2+P are connected sets. Denote the set of points common to k and M_i $(i=1, 2)$ by k_i. Neither of these sets is vacuous, since both M_1 and M_2 have points in common with t, and hence with k. Clearly $k-P$ is the sum of the two mutually separated sets k_1 and k_2. That is, P is a cut-point of k.

It follows by the theorem of Knaster and Kuratowski referred to above that k_1+P and k_2+P are connected sets. If x is any point of M_1 interior to T, then x is a point of k and a fortiori of k_1. Then there exists a connected subset of M_1+P, namely k_1+P, which contains both x and P and lies wholly within K. That is, the set of all points of M_1+P which lie interior to T lie, with P, in a connected subset of M_1+P which lies wholly interior to K. Hence M_1+P is regular at P. That it is regular at all other points is easily seen. Similarly, M_2+P is regular.

THEOREM 6. *If M is a regular point set, R a region of M, and P a point of R, then if k is a maximal connected subset of $R-P$, $k+P$ is a regular connected point set.*

PROOF. If $R-(k+P)$ is vacuous, $k+P$ is a regular connected set by Theorem 2. If $R-(k+P)$ is not vacuous, denote it by q. Then $R-P$ is the sum of the two mutually separated sets k and q, and hence, by Theorem 5, $k+P$ is connected and regular.

DEFINITION. If M is a point set, and C_1 and C_2 are mutually exclusive point sets, and H is a connected subset of M which has no point in common with either C_1 or C_2, but has limit points which are points of M in both C_1 and C_2, then H, together with those points of M in C_1+C_2 which are limit points of H, will be called a set $K(C_1, C_2)M$.† If the set

* B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255, Theorem 6.

† I have made use of the sort of set defined here in other connections. See my paper *A property which characterizes continuous curves*, Proceedings of the National Academy, vol. 11 (1925), pp. 725–728. Also see this Bulletin, vol. 32 (1926), p. 218, paper 4.

H is identical with that maximal connected subset of $M-M\times C_1-M\times C_2$ determined by H, then H together with those points of M in C_1+C_2 which are limit points of H, will be called a *maximal set* $K(C_1,C_2)M$ or, for the sake of brevity, a set $K'(C_1, C_2)M$.

DEFINITION. If M is a regular point set, P a point of M and C a circle enclosing P, then a *branch of* M *with respect to* P *and* C is a set $K'(P, C)M$.

THEOREM 7. *There does not exist, in the plane, a regular, connected, unbounded point set which remains connected upon the omission of any bounded connected subset.*

PROOF. Suppose there does exist such a set. Denote it by M.

1. If C is any circle, and P a point of M within C, say for convenience at the center of C, then I shall show that there exist infinitely many distinct branches of M with respect to P and C such that any two of these are, except for P, mutually separated.

Let S be the region of M which is determined by P and C. Since M is regular, there exists a circle T with center at P such that all points of M interior to T lie in S. Denote the set of all points of M interior to T by T_1.

The set $S-P$ is not connected. For if it were, then $M-(S-P)$ would, by hypothesis, be connected, which is impossible since this set contains no point of T_1 except P. Then $S-P$ is the sum of two mutually separated sets S_1 and S_2. The sets S_1+P and S_2+P are connected, and by Theorems 2 and 5 are regular. If x is a point of S_1, then by Theorem 4 the set S_1+P contains a set $K(P, x)$ (S_1+P). Denote this set by k_1. The set $M-P$ being unbounded, connected (by hypothesis) and regular, contains a set $K(x, C)$ $(M-P)$. Call this set k_2. Denote that portion of k_2 which is not on C by k_2'. The set k_2' is connected and, since it contains x, is a subset of S_1. Denote the set k_1+k_2 by k. Then k is a set $K(P, C)M$. If to the set k_1+k_2 be added all those points of that maximal connected subset of $S-P$ determined by x, as well as the limit points of these points which are in M and on C, the resulting set is a set $K'(P, C)M$, which will be denoted by K_1. Denote the set of points $K_1-P-K_1\times C$ by H_1. Then H_1 is a connected subset of S_1.

In a similar way it can be shown that there exists a set H_2 which is a subset of S_2, and which, together with P and its limit points on C that belong to M, forms a set $K'(P, C)M$ which will be denoted by K_2. The sets H_1 and H_2 are mutually separated.

(a) If K_1-P and K_2-P are not mutually separated, their sum forms a connected set N.

(b) If K_1-P and K_2-P are mutually separated, let the set of all points of K_i $(i=1, 2)$ which lie between and on the circles C and T be denoted by B_i. The sets B_1 and B_2 are closed in $M-P$ and mutually exclusive. Hence, by Theorem 4, there exists a set $K(B_1, B_2)$ $(M-P)$ which is bounded. Denote this set by N_1. Then $(K_1-P)+(K_2-P)+N_1$ is a bounded connected subset of $M-P$ which will be denoted by N.

In either case (a) or (b), then, there exists a circle G concentric with C, whose radius is greater than the radius of C, and which encloses a connected subset, N, of $M-P$, which contains K_1-P and K_2-P. If g is the region of M determined by P and G, then $g-P$ is the sum of two mutually separated sets g_1 and g_2, and g_1+P and g_2+P are connected and regular. As $N+P$ is connected and lies within G, it is clear that it is a subset of g. Hence N is a subset of g_1+g_2 and being connected is a subset of one of the sets g_1, g_2, say g_1. As above, we can show that M contains a set k_3 which is a branch of M with respect to P and G, and which, except for P and its points on G, is a maximal connected subset of g_2. If h_3 denotes that portion of k_3 which is not on G, then the set h_3+P is regular, by Theorem 6, and hence contains a set $K'(P, C)M$ which will be denoted by K_3. It is clear that the sets K_2 and K_3 are, except for P, mutually separated. Denote the set $K_3-P-K_3\times C$ by H_3.

By means of Theorem 4, it can be shown that there exists a circle G_1 of radius greater than the radius of G, such that the sets $K_i - P$ $(i=1, 2, 3)$ all lie in a connected subset, F, of M, which lies within G_1. If q is the region of M determined by P and G_1, then $q-P$ is the sum of two mutually separated sets, q_1 and q_2, one of which, say q_1, contains F. Then a subset K_4 of q_2+P may be found which is a set $K'(P, C)M$.

Continuing as indicated above, the existence of an infinite sequence of sets $K_2, K_3, K_4, \cdots$, such that for every positive integer $n>1$, K_n is a set $K'(P, C)M$, and such that any two of these sets are, except for P, mutually separated, is established.

2. Consider in particular the sets K_2, K_3, K_4, and K_5. For each i, $(i=2, 3, 4, 5)$, let A_i be a point of K_i on C. Two of these points must separate the other two on C; say A_2 and A_3 separate A_4 and A_5 on C. As $M-(K_4+K_5)$ is connected, by hypothesis, and regular since K_4+K_5 is closed in M, it follows that there exists, by Theorem 4, a bounded set $K(A_2, A_3)$ $(M-K_4-K_5)$ which, together with the set of points K_2+K_3-P, is a bounded connected subset V of $M-P$. There exists a circle E concentric with C, which encloses V and contains no limit points of it.

Just as the existence of branches of M with respect to P and C were established, it can be shown that there exists an infinite set $K_i(A_j)$, $(i=1, 2, 3, \cdots, j=4, 5)$, of branches of M with respect to A_j and E. From the definition of a branch of M, it is clear that the connected set $V+K_4+K_5$ must lie wholly in one branch of M with respect to A_4 and E, say in $K_1(A_4)$, and in one branch of M with respect to A_5 and E, say $K_1(A_5)$.

As the set V contains no limit points of the set of points $K_4+K_5+K_2(A_4)+K_2(A_5)$, every point of it is the center of a circle which neither encloses any point of the latter set nor of E, nor has any point in common with either. The sum of the interiors of all such circles is a connected domain and this domain contains an arc t_1 from A_2 to A_3. Clearly t_1 and $K_4+K_5+K_2(A_4)+K_2(A_5)$ are mutually separated.

The points sets K_2+K_3 and $K_2(A_4)+K_2(A_5)$ are mutually separated. Denote by U the set of points consisting of $K_2(A_4)+K_2(A_5)$ together with its limit points. Then K_2+K_3 and U are mutually exclusive. Let $C(A_2)$ and $C(A_3)$ be circles with centers at A_2 and A_3, respectively, and enclosing no point of U. If a_i $(i=2, 3)$ is a point of H_i lying within $C(A_i)$, there exists an arc b_i joining A_i and a_i which lies entirely within $C(A_i)$, and except for A_i lies wholly within C. As H_2+H_3+P is a connected subset of C containing a_2 and a_3 but no point of U, there exists, by Theorem H of my paper *On a certain type of connected set which cuts the plane*,* an arc b_1 which joins a_2 and a_3, contains no point of U, and lies wholly within C. Clearly the continuous curve consisting of the arcs b_1, b_2, and b_3 contains an arc t_2 which joins A_2 and A_3, lies except for these points entirely within C, and contains no point of U. Similarly, since K_4+K_5 and t_1 are mutually separated, there exists an arc t_3 joining A_4 and A_5, lying except for these points entirely within C, and having no point in common with t_1.

By the corollary to Theorem D of the paper referred to in the preceding paragraph,† there exists a simple closed curve J which is a subset of t_1+t_2 and separates the plane between A_4 and A_5. However, J has no points in common with either $K_2(A_4)$, $K_2(A_5)$ or E, and yet the sum of these three sets is a connected set containing A_4 and A_5. Thus a contradiction is established and the theorem is proved.

THE UNIVERSITY OF MICHIGAN

* To appear in the Proceedings of the International Mathematical Congress at Toronto. Theorem H of this paper is the following: Let G be a bounded domain, K any closed set of points and N a connected subset of G which contains no points of K. Then every pair of distinct points of K are the end-points of an arc which lies in G and contains no points of K.

† The corollary referred to here is the following: If A and B separate C and D on a simple closed curve K, AB and CD are arcs joining A,B and C,D, respectively, and lying, except for their end-points, interior to K, and t is an arc from A to B that contains no points of CD, then there exists a simple closed curve J which is a subset of $AB+t$, such that C is interior to J and D exterior to J, or vice versa.