THE ASYMPTOTIC OSCULATING QUADRICS OF A CURVE ON A SURFACE

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1. *Definition.* Let us consider a surface S, a curve C on S, and three neighboring points P, P_1, P_2, on C. The three tangents at these points to the asymptotic curves of one family determine a quadric whose limit, as P_1, P_2 independently approach P along C, is a quadric called* by Bompiani an *asymptotic osculating quadric* of C at P. A second asymptotic osculating quadric is obtained by using the other family of asymptotics. We shall now derive the equations of these quadrics, using Wilczynski's notation, and shall deduce some of their fundamental properties.

2. *Equations.* Let the four homogeneous coordinates y of a point P on a non-degenerate non-developable surface S be given as analytic functions of two independent variables u, v; and let the curves $u=\text{const.}$, $v=\text{const.}$ be the asymptotics. Then the functions y, when multiplied by a suitably chosen proportionality factor, are solutions of Wilczynski's canonical system of differential equations,

$$y_{uu} + 2by_v + fy = 0, \quad y_{vv} + 2a'y_u + gy = 0. \tag{1}$$

The one-parameter family of curves on S represented by the equation

$$dv - \lambda du = 0 \tag{2}$$

contains one curve C through P. The coordinates Y of any point P_1 on C near P are given by an expansion of the form

$$Y = y + \frac{dy}{du}\Delta u + \frac{1}{2}\,\frac{d^2y}{du^2}\Delta u^2 + \cdots .$$

* Bompiani, *Geometria delle superficie considerate nello spazio rigato*, Rendiconti dei Lincei, 1926.

If the points y, y_u, y_v, y_{uv} are used as the vertices of a local tetrahedron of reference with a suitably chosen unit point, the local coordinates x_i of P_1 are represented by the series

$$
(3) \quad \left\{
\begin{aligned}
x_1 &= 1 - \frac{1}{2}(f + g\lambda^2)\Delta u^2 + \left[-\frac{1}{6}f_u + \frac{1}{2}(2bg - f_v)\lambda \right. \\
&\quad \left. + \frac{1}{2}(2a'f - g_u)\lambda^2 - \frac{1}{6}g_v\lambda^3 - \frac{1}{2}g\lambda\lambda' \right]\Delta u^3 + \cdots, \\
x_2 &= \Delta u - a'\lambda^2\Delta u^2 + \left[-\frac{1}{6}f + 2a'b\lambda \right. \\
&\quad \left. - \frac{1}{2}(g + 2a_u')\lambda^2 - \frac{1}{3}a_v'\lambda^3 - a_u'\lambda\lambda' \right]\Delta u^3 + \cdots, \\
x_3 &= \lambda\Delta u - \left(b - \frac{1}{2}\lambda'\right)\Delta u^2 + \left[-\frac{1}{3}b_u \right. \\
&\quad \left. - \frac{1}{2}(f + 2b_v)\lambda + 2a'b\lambda^2 - \frac{1}{6}g\lambda^3 + \frac{1}{6}\lambda'' \right]\Delta u^3 + \cdots, \\
x_4 &= \lambda\Delta u^2 - \frac{1}{3}\left(a'\lambda^3 + b - \frac{3}{2}\lambda'\right)\Delta u^3 + \cdots,
\end{aligned}
\right.
$$

where λ', λ'' are total derivatives of λ. The derivatives of these series with respect to Δu are the coordinates x_{iu} of a point on the tangent of the asymptotic $v = \text{const.}$ through P_1. And any point on this tangent is given by a linear combination of the form

$$\eta_i = hx_i + kx_{iu}.$$

If now the algebraic equation of a quadric surface is subjected to the condition that it be satisfied by the functions η identically in h, k and in Δu up to and including terms in Δu^2, the result is

$$
(4) \quad \left\{
\begin{aligned}
&(2a'b\lambda^3 - 2b_v\lambda^2 - b_u\lambda + 2b^2 + b\lambda')x_4^2 + 2b\lambda x_3x_4 \\
&\quad + \lambda^3(x_1x_4 - x_2x_3) - 2b\lambda^2x_2x_4 = 0.
\end{aligned}
\right.
$$

This is the equation of the first asymptotic osculating quadric $Q^{(u)}$ of C at P. The equation of the second asymptotic osculating quadric $Q^{(v)}$ of C at P is

$$
(5) \quad \left\{ \begin{aligned} &(2a'b - 2a_u'\lambda - a_v'\lambda^2 + 2a'^2\lambda^3 - a'\lambda')x_4^2 - 2a'\lambda x_3 x_4 \\ &\quad + (x_1x_4 - x_2x_3) + 2a'\lambda^2 x_2 x_4 = 0. \end{aligned} \right.
$$

3. *Properties.* Some simple properties of the quadrics $Q^{(u)}$ and $Q^{(v)}$ will now be deduced. First of all, it is clear that the tangent plane $x_4 = 0$ of S at P cuts each of these quadrics in the asymptotic tangents $x_2x_3 = 0$. And $Q^{(u)}$ becomes the quadric Q of Lie, whose equation is

$$
(6) \qquad x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0,
$$

in case $\lambda \to \infty$, while $Q^{(v)}$ becomes the quadric of Lie in case $\lambda = 0$. The quadrics $Q^{(u)}$ and $Q^{(v)}$ coincide only for a curve C which is tangent to a curve of Darboux, $a'\lambda^3 + b = 0$, on a surface for which

$$
a'\left[\frac{\partial}{\partial u}\log a'^2 b\right]^3 = b\left[\frac{\partial}{\partial v}\log a' b^2\right]^3.
$$

We shall suppose from now on that Q, $Q^{(u)}$, $Q^{(v)}$ are distinct, and that S is unrestricted.

The result of eliminating λ' from equations (4) and (5) is

$$
(7) \quad \left\{ \begin{aligned} &(a'\lambda^3 + b)(x_1x_4 - x_2x_3 + 2a'bx_4^2) \\ &+ a'b\left[2(a'\lambda^3 + b) - \lambda\frac{\partial}{\partial u}\log a'^2 b \right. \\ &\left. - \lambda^2\frac{\partial}{\partial v}\log a'b^2\right]x_4^2 = 0. \end{aligned} \right.
$$

This is a quadric of Darboux through the intersection of $Q^{(u)}$ and $Q^{(v)}$. For a curve C which passes through P tangent to a curve of Darboux this quadric becomes the tangent plane counted twice, so that the intersection of $Q^{(u)}$ and $Q^{(v)}$ is the asymptotic tangents each counted twice; in this case $Q^{(u)}$ and $Q^{(v)}$ are tangent to each other at every point of each asymptotic tangent.

We have found here a new characterization of the directions of Darboux: *the directions of Darboux are the di-*

*rections of curves whose asymptotic osculating quadrics intersect only in the asymptotic tangents.**

The quadrics $Q^{(u)}$ and $Q^{(v)}$ intersect, besides in the asymptotic tangents, also in a residual conic which lies in the plane whose equation is

$$(8)\quad \begin{cases} 2\lambda(a'\lambda^3 + b)(x_3 - \lambda x_2) + [\lambda'(a'\lambda^3 + b) - 2(a'^2\lambda^6 - b^2) \\ \quad + \lambda(2a_u'\lambda^3 - b_u) + \lambda^2(a_v'\lambda^3 - 2b_v)]x_4 = 0, \end{cases}$$

obtained by eliminating x_1 from equations (4) and (5). This plane cuts the tangent plane $x_4 = 0$ in the line $x_3 - \lambda x_2 = 0$, which is tangent to C at P; and the residual conic has this line for tangent at P.

The equation of the osculating plane of C at P is

$$(9)\qquad 2\lambda(x_3 - \lambda x_2) - (\lambda' + 2a'\lambda^3 - 2b)x_4 = 0.$$

This plane coincides with the plane (8) in case C is such that

$$(10)\quad 2(a'\lambda^3 + b)\lambda' + \lambda(2a_u'\lambda^3 - b_u) + \lambda^2(a_v'\lambda^3 - 2b_v) = 0.$$

But this is the differential equation† of the pangeodesics. Thus we have found a new characterization of the pangeodesics: *a curve is a pangeodesic in case its osculating plane contains the residual conic of intersection of its asymptotic osculating quadrics.*

The quadric of Lie intersects $Q^{(u)}$ in the asymptotic tangents and in a conic which lies in the plane

$$(11)\quad 2b\lambda(x_3 - \lambda x_2) + (b\lambda' + 2b^2 - b_u\lambda - 2b_v\lambda^2)x_4 = 0.$$

This plane coincides with the osculating plane of C in case C is such that

$$(12)\qquad 2b\lambda' + 2a'b\lambda^3 - 2b_v\lambda^2 - b_u\lambda = 0.$$

The importance of the pole-polar correspondence with respect to the quadric of Lie in the projective differential geometry of surfaces suggests that the polar relation with respect to the quadrics $Q^{(u)}$ and $Q^{(v)}$ would be of interest.

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* This theorem was also found by Bompiani. See Fubini and Čech, *Geometria Proiettivo Differentiale*, vol. 2, Appendix 2.

† Fubini and Čech, *Geometria Proiettiva Differenziale*, p. 147.

We shall merely mention here that a curve C defines by means of these quadrics a collineation in the tangent plane such that to any point x corresponds that point x' which has the same polar plane with respect to $Q^{(v)}$ as the point x has with respect to $Q^{(u)}$. The equations of this collineation are

$$(13) \quad \begin{cases} \rho x_1' = \lambda^2 x_1 + 2(a'\lambda_3 + b)(x_3 - \lambda x_2), \\ \rho x_2' = \lambda^2 x_2, \qquad \rho x_3' = \lambda^2 x_3. \end{cases}$$

This collineation is an *elation*, with the tangent of C for axis and with the point P for center. It is the identity if C is tangent to a curve of Darboux.

4. *Conjugate Asymptotic Osculating Quadrics.* The family of curves

$$(14) \qquad dv + \lambda du = 0$$

is conjugate to the family (2). The curve $C_{-\lambda}$ of this family through P is conjugate to the curve C_λ of the family (2) through P, so that the corresponding quadrics $Q_\lambda{}^{(u)}$, $Q_{-\lambda}{}^{(u)}$ and $Q_\lambda{}^{(v)}$, $Q_{-\lambda}{}^{(v)}$ may be called conjugate. The equations of conjugate quadrics differ only in the sign of λ.

It is easy to verify that $Q_\lambda{}^{(u)}$ and $Q_{-\lambda}{}^{(u)}$, intersecting in the asymptotic tangents, are tangent to each other along the line $x_2 = x_4 = 0$. The remainder of their intersection is the straight line

$$(15) \quad \begin{cases} x_2 - \left(\dfrac{1}{2}\dfrac{\lambda_v}{\lambda} - \dfrac{b_v}{b} + \dfrac{b}{\lambda^2}\right)x_4 = 0, \\ x_1 - \left(\dfrac{1}{2}\dfrac{\lambda_v}{\lambda} - \dfrac{b_v}{b} - \dfrac{b}{\lambda^2}\right)x_3 \\ \qquad\qquad + \left(2a'b - \dfrac{b_u}{\lambda^2} + b\dfrac{\lambda_u}{\lambda^3}\right)x_4 = 0, \end{cases}$$

which intersects the asymptotic tangent $x_2 = x_4 = 0$ in the point

$$\left(-\frac{b}{\lambda^2} + \frac{1}{2}\frac{\lambda_v}{\lambda} - \frac{b_v}{b}, 0, 1, 0\right).$$

Similarly, the quadrics $Q_\lambda^{(v)}$ and $Q_{-\lambda}^{(v)}$ define the point

$$\left(-a'\lambda^2 - \frac{1}{2}\frac{\lambda_u}{\lambda} - \frac{a_u'}{a'},1,0,0\right)$$

on the other asymptotic tangent. The line joining these two points coincides with the ray of the conjugate net λ, which joins the points

$$\left(-\frac{1}{2}\frac{\lambda_v}{\lambda} - \frac{b}{\lambda^2},0,1,0\right), \qquad \left(\frac{1}{2}\frac{\lambda_u}{\lambda} - a'\lambda^2,1,0,0\right),$$

in case

$$\frac{\lambda_u}{\lambda} + \frac{a_u'}{a'} = 0, \qquad \frac{\lambda_v}{\lambda} - \frac{b_v}{b} = 0.$$

Then the surface S has the property

$$\frac{\partial^2}{\partial u \partial v} \log a'b = 0,$$

so that, after a change of parameters, $a'b=1$, $\lambda = \text{const.}\ b$. For such surfaces Wilczynski's canonical form (1) of the differential equations coincides with Fubini's canonical form.* The line y, y_{uv} is therefore the projective normal. The curvature of Fubini's fundamental form $\phi_2 \equiv 8a'b\, du\, dv$ is zero, and the mean projective curvature† is -2.

We reach thus the following conclusion. *A surface has mean projective curvature equal to* -2 *if, and only if, there exists on it a conjugate net whose ray, for each surface point, coincides with the line joining the points on the asymptotic tangents where each of these tangents is met by the residual line of intersection of a pair of conjugate asymptotic osculating quadrics of the curves of the conjugate net through the surface point.*

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* Lane, *Wilczynski's and Fubini's canonical systems of differential equations*, this Bulletin, vol. 32 (1926), p. 365.

† Fubini and Čech, loc. cit., p. 146.