GENERALIZATION OF LAGRANGE'S THEOREM

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1. *Introduction.* The following theorem due to Lagrange is of considerable importance in the theory of equations.

LAGRANGE'S THEOREM. *If the group to which the rational function* $\psi(x_1, \cdots, x_n)$ *belongs is a subgroup of the group to which the rational function* $\phi(x_1, \cdots, x_n)$ *belongs, then* ϕ *equals a rational function of* ψ *and the elementary symmetric functions of the variables* $x_1, \cdots, x_n$.

In this paper I prove a similar theorem for sets of variables.

2. *Notation and Definitions.* Consider the n sets of m variables $x_{1i}, x_{2i}, \cdots, x_{mi}$ $(i=1, \cdots, n)$, which may be regarded as coordinates of n points in m-space. By a permutation of these sets of variables we mean a permutation of the points. Thus a permutation which changes x_{1i} to x_{1j}, also changes $x_{2i}, \cdots, x_{mi}$ to $x_{2j}, \cdots, x_{mj}$ respectively. It is simpler to regard the permutation as affecting the second subscripts of the variables, with the above notation, than as affecting the x's.

A function $\phi(x_{11}, x_{21}, \cdots, x_{m1}; \cdots; x_{1n}, x_{2n}, \cdots, x_{mn})$ is said to *belong* to a substitution group G on the symbols $1, 2, \cdots, n$, if ϕ is unaltered by every substitution of G and by no substitution on these symbols not contained in G. There exist functions which belong to a given substitution group. In fact, we can construct such functions involving only the variables $x_{11}, x_{12}, \cdots, x_{1n}$.*

3. *A Generalization.* We proceed to prove the following generalization of Lagrange's Theorem.

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* Netto, *Substitutionentheorie und ihre Anwendung auf die Algebra*, 1882, p. 27.

THEOREM. *If the group to which the rational function* $\psi(x_{11}, x_{21}, \cdots, x_{m1}; \cdots; x_{1n}, x_{2n}, \cdots, x_{mn})$ *belongs, is a subgroup of the group to which the rational function* $\phi(x_{11}, x_{21}, \cdots, x_{m1}; \cdots; x_{1n}, x_{2n}, \cdots, x_{mn})$ *belongs, then* ϕ *equals a rational function of* ψ *and the elementary symmetric functions of the sets of variables* $x_{1i}, x_{2i}, \cdots, x_{mi}$, $(i=1, \cdots, n)$.

It will suffice to consider the case $m=3$. The elementary symmetric functions of the n triads of variables are defined by*

$$p_{ijk}=\sum x_{11}x_{12}\cdots x_{1i}\, x_{2,1+i}x_{2,2+i}\cdots x_{2,j+i}x_{3,1+i+j}x_{3,2+i+j}\cdots x_{3,k+i+j} \qquad (i+j+k\leqq n).$$

With the aid of these functions, we can express any one of the variables x_{1i}, x_{2i}, x_{3i} as a rational function of any one of the others. In fact,† we have

$$x_{1i}=\frac{p_{100}x_{3i}^{n-1}-p_{101}x_{3i}^{n-2}+p_{102}x_{3i}^{n-3}-\cdots}{nx_{3i}^{n-1}-(n-1)p_{001}x_{3i}^{n-2}+(n-2)p_{002}x_{3i}^{n-2}-\cdots},$$

$$x_{2i}=\frac{p_{010}x_{3i}^{n-1}-p_{011}x_{3i}^{n-2}+p_{012}x_{3i}^{n-3}-\cdots}{nx_{3i}^{n-1}-(n-1)p_{001}x_{3i}^{n-2}+(n-2)p_{002}x_{3i}^{n-3}-\cdots}.$$

Hence every function of the triads of variables can be expressed as a function of $x_{31}, x_{32}, \cdots, x_{3n}$, with coefficients that belong to the symmetric group. In particular, suppose

$$\psi(x_{11},x_{21},x_{31};\cdots;x_{1n},x_{2n},x_{3n}) = \psi_1(x_{31},x_{32},\cdots,x_{3n}),$$

$$\phi(x_{11},x_{21},x_{31};\cdots;x_{1n},x_{2n},x_{3n}) = \phi_1(x_{31},x_{32},\cdots,x_{3n}).$$

Evidently ψ and ψ_1 belong to the same group H, and ϕ and ϕ_1 belong to the same group G. As H is a subgroup of G by hypothesis, it follows from Lagrange's Theorem, that ϕ equals a rational function of ψ and the elementary symmetric functions $p_{001}, p_{002}, \cdots, p_{00n}$. The theorem follows.

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* See Bôcher, *Higher Algebra*, p. 252.

† Netto, *Vorlesungen über Algebra*, vol. II, p. 71.