THE THIRTY-SECOND SUMMER MEETING OF THE SOCIETY

The thirty-second summer meeting of the American Mathematical Society was held at Ohio State University, Columbus, Ohio, on Wednesday and Thursday, September 8-9, 1926, immediately preceded by the summer meeting of the Mathematical Association of America. A joint session of the two organizations was held on Wednesday morning, at which addresses were delivered by Professor E. W. Chittenden on *The metrization problem and related problems in the theory of abstract sets*, and Professor E. T. Bell on *Successive generalizations in the theory of numbers*. Professor Chittenden's address will be published in an early issue of this BULLETIN; Professor Bell's will appear in the AMERICAN MATHEMATICAL MONTHLY. There was a joint dinner of the two organizations on Wednesday evening, at the Faculty Club of the University. At the close of the sessions a resolution was passed expressing the appreciation of the Society for the very cordial hospitality of the University, which made this meeting one of the pleasantest in its history. The authorities of the Ohio State University contributed greatly to the success of this meeting by putting the dormitories, Mack Hall and Oxley Hall, at the disposal of the Society; the reception held on Tuesday evening in the Faculty Club was a very pleasant occasion. The cordiality of the officials of the University and the unremitting zeal of the members of its Department of Mathematics to promote the success of the meeting will be gratefully remembered by all those who were present.

There was a large attendance, probably the largest for any summer meeting at which no colloquium was held. It included the following one hundred seventeen members of the Society.

Akers, W. E. Anderson, Anning, Arnold, Bareis, Barnett, E. R. Beckwith, E. T. Bell, H. A. Bender, S. R. Benedict, A. A. Bennett, Blumberg, Borger, Bowles, Bussey, Cairns, C. C. Camp, J. W. Campbell, Candy, Caris, Carmichael, Chang, Chittenden, E. H. Clarke, Abraham Cohen, L. P. Copeland, Cramblet, Crane, Dantzig, Davisson, J. M. Davis, Dresden, Dushnik, Emch, G. C. Evans, Everett, Farnum, Feinler, Peter Field, Finkel, Fite, Foraker, W. B. Ford, Fry, C. A. Garabedian, Garretson, D. C. Gillespie, Glenn, B. C. Glover, J. W. Glover, Gummer, Harkin, E. R. Hedrick, Hollcroft, Ingraham, Dunham Jackson, Kempner, Kuhn, LeStourgeon, A. D. Lewis, F. A. Lewis, Lindemann, MacDuffee, T. E. Mason, A. S. Merrill, Michal, Michie, G. A. Miller, I. L. Miller, Mills, Miser, C. L. E. Moore, C. C. Morris, F. H. Murray, Newlin, Olson, F. W. Owens, Pennell, Rasor, Raynor, R. G. D. Richardson, Rider, H. L. Rietz, J. C. Rietz, E. D. Roe, J. R. Roe, Rowland, Seely, Shewhart, Shohat, W. G. Simon, Sisam, Slaught, Smail, C. E. Smith, Spenceley, Stafford, Swartzel, Teach, T. Y. Thomas, Tinner, Tripp, B. M. Turner, Vandiver, T. O. Walton, J. H. Weaver, R. A. Wells, Westfall, R. L. Wilder, Wiley, C. O. Williamson, F. M. Wright, Yanney, Yeaton, J. W. Young, Yowell, Zehring.

The Council announced the election to patron membership of

The Bell Telephone Laboratories;

and to sustaining membership of

Mr. Adolph S. Ochs.

It was announced that since the April meeting of the Council and at this meeting the following thirty-three persons were elected to ordinary membership in the Society.

Miss Nola Lee Anderson, Saint Catharine, Missouri;
Mr. Thomas C. Benton, University of Pennsylvania;
Mr. William J. Berry, University of Colorado;
Mr. Edward Bright, Sandwich, Cape Cod, Mass.;
Mr. John Clark Brixey, University of Oklahoma;
Mr. Bailey LeFevre Brown, Princeton University;
Mr. William Bucke Campbell, Cornell University;
Professor Harry Clyde Carver, University of Michigan;
Mr. Frank Hsi-Lu Chang, University of Chicago;
Dr. Edgar Dehn, Wagner College, Staten Island, New York;
Dr. Frederick William Doermann, New York University;
Mr. Howard P. Doole, University of Wisconsin;
Mr. Leamon A. Dye, University of Rochester;
Miss Fay Farnum, Cornell University;
Professor John Wildeboor Hurst, Montana State College;
Mr. Maurice L. Hartung, University of Wisconsin;
Professor William Scribner Kimball, Michigan State College;
Professor Jean Martial Lapeyre, Loyola University, New Orleans;
Mr. Derrick Henry Lehmer, University of California;

Mr. Clarence Isaac Lubin, University of Cincinnati;
Mr. Charles Thomas Male, Union College;
Mr. Alexander Maslow, University of Michigan;
Mr. John M. Melchiors, 510 Sixth St., Kaukauna, Wis.;
Mr. Henry Otten, Jr., Interborough Rapid Transit Co., New York, N. Y.
Mr. Harry S. Pollard, University of Wisconsin;
Mr. Harold Conway Shaub, Cornell University;
Professor James Abram Garfield Shirk, Kansas State Teachers College;
Professor Emory Potter Starke, Rutgers University;
Mr. Earl Edward Steinert, Union College;
Mr. George Clarence Vedova, Columbia University;
Professor Lloyd A. H. Warren, University of Manitoba;
Mr. Gordon T. Whyburn, University of Texas;
Mr. John Williamson, University of Chicago.

The following person was elected to membership as nominee of the New York Edison Company:

Mr. J. W. Lieb.

The secretary announced that the following persons had taken membership in the Society under the reciprocity agreement with the London Mathematical Society:

Professor T. Kubota, Tôhoku Imperial University;
Dr. O. Szasz, Frankfurt a/M., Germany.

Eighteen applications for membership were received.

It was announced that the following persons had been appointed to represent the Society: on the Sub-Committee on Mathematics and Mathematical Signs of the Sectional Committee on Scientific and Engineering Symbols and Abbreviations of the American Engineering Standards Committee, Professors Abraham Cohen and Oswald Veblen; to represent the Society at the Seventy-Fifth Anniversary of the founding of Milwaukee College on June 12, 1926, Professor Arnold Dresden; to represent the Society at the Semi-Centennial of Grove City College on June 15–16, 1926, Professor O. P. Akers; to represent the Society at the Centennial of Western Reserve University on November 12–13, 1926, Professor W. G. Simon; to represent the Society at the inauguration of President A. B. Hall at the University of Oregon on October 18–23, Professor F. L. Griffin. Also that a Committee on policy had been appointed to consist of

Professors Bliss, Eisenhart, Huntington, Dunham Jackson, and R. L. Moore.

Further announcements included that President Birkhoff's retiring address to be delivered at the annual meeting in Philadelphia is entitled *A mathematical critique of some physical theories*; that the 1927 meetings of the Society in New York have been set for the last Saturday in February, the last Saturday in October, and the first Saturday in May; that the Summer Meeting and Colloquium for 1927 are to be held at Madison, Wisconsin, during the week beginning September 4.

The Council accepted the invitations of Amherst College and of the University of Colorado to hold the summer meetings of the Society at these institutions in 1928 and 1929 respectively.

At the joint session on Wednesday forenoon the following resolution was adopted:

"*Resolved*, That the American Mathematical Society and the Mathematical Association of America, in joint session at Columbus on September 8, 1926, express their deep sense of loss in the death of Professor Bohannan, on June 20, 1926, and their profound regret that the meetings on the campus so long animated by his personality could not be graced by his presence; and direct that copies of this resolution be transmitted to Mrs. Bohannan and to the President of the Ohio State University."

The following resolution was adopted at the session of Wednesday afternoon:

"On recommendation of its Council, the American Mathematical Society, in session at Columbus, Ohio, on September 9, 1926, expresses its deep sense of loss in the death of Frank Nelson Cole on May 26, 1926. He was for many years the Society's most active executive officer. From an early date in its history until 1920, when he passed his duties on to others, he ably guided the development of the Society. As Secretary from 1895, as a member of the Editorial Board of the BULLETIN from 1898, and as its Editor-in-Chief from 1900, he led the Society from its modest beginnings to a state of solid accomplishment. He exercised his functions with a skill which excited admiration and which gave the American Mathematical Society an established place in the scientific world. When he retired, he could turn over to his successors a healthy structure, which was able to withstand the stresses of the very difficult post-war period. The Society wishes also to place on record at this time its grateful recognition of his devoted service to the ideals of American mathematical science. His memory will remain an inspiration to all who may in the future serve the interests of the Society and the cause of mathematics in America."

Professor G. C. Evans, vice-president of the Society, presided at the beginning of the joint session on Wednesday morning, relieved by Professor Dunham Jackson, president of the Association. The later sessions of the Society were presided over by Vice-Presidents Evans and E. T. Bell and Professor R. D. Carmichael.

Titles and abstracts of the papers read at the regular sessions follow below. The papers of Aldrich, Altshiller-Court, Dickson, Ettlinger, Evans, Gehman, Griffiths, Hazlett, Hill, Kellogg, Koopman, Michal, Norman Miller, Morse, Murnaghan, Neelley, Niemytski, Rainich, Rees, Robinson, Sierpinski, Snyder, Stone, Sturdivant, J. M. Thomas, T. Y. Thomas and Michal, Vandiver (second paper), Vandiver and Wahlin, Vinogradov, Walsh, G. T. Whyburn, W. M. Whyburn and C. E. Wilder were read by title. Mr. Mullings, Mr. Rees, and Mr. Sturdivant were introduced by Professor Ettlinger, Professor Vinogradov by Professor H. H. Mitchell, and Professors Niemytski and Sierpinski by Professor Chittenden.

1. Professor J. H. Weaver: *Invariants of a poristic system of triangles.*

Gallatley in his book *Modern Geometry of the Triangle* discusses certain properties of a system of triangles having a fixed incircle and a fixed circumcircle. However he does not attempt to find what curves associated with the system remain invariant. The author shows that there is a single invariant line associated with the system of triangles, namely, the line determined by the points where the bisectors of the exterior angles meet the opposite sides. He also shows that there is a one-parameter family of invariant curves of the second degree associated with the system.

2. Professor T. R. Hollcroft: *Consecutive multiple points of algebraic curves.*

Of the total number of conditions on an algebraic curve accounted for by a given series of consecutive multiple points, some are projective and some metric. The number of conditions of each kind can be found when the constituents and the manner of formation of the singularity are known. In the determination of a curve, projective conditions belonging to independent singularities are always independent and necessary while metric conditions associated with any singularity are not necessary and any or all of them may be dependent. For a given singularity, it is sufficient but not necessary that the multiple points which became consecutive to build up this singularity be the same as those into which it can be resolved. There is a definite number of projective as well as a definite number of metric conditions belonging to a given singularity. The number of projective conditions actually given by a singularity is always this number however the singularity may have been formed, but the number of metric conditions varies with the manner of formation. Limits are derived for the number of consecutive multiple points of any order, with or without coincident branches, on curves of any order.

3. Professor F. A. Foraker: *Nepolar locus of a point with respect to a conic.*

If A, P, B, and Q are four points on a line which satisfy the condition $(AP/PB)\cdot(AQ/QB)=-1$ they will be defined as a neharmonic range. The nepolar locus of a point with respect to a conic is the locus of the neharmonic conjugate of the given point with respect to the two points in which any secant through the given point cuts the conic. The nepolar locus of the point (x_1, y_1) with respect to the conic $F(x, y)=ax^2+2hxy+by^2+2gx+2fy+c=0$ is the cubic $FP-2FF_1+F_1P=0$, where $p=axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c$. It is shown that the nepolar locus of a point with respect to a conic passes through the points of intersection of the conic and its polar with respect to the conic. Also, that the nepolar locus of a point on a conic with respect to that conic itself consists of the conic and its tangent at the given point. A number of graphs are shown exhibiting a variety of cubics derived from the equation.

4. Professor G. Y. Rainich: *On singularities of vector fields in space.*

A configuration consisting of a closed surface with a vector in every point of it is studied from the topological point of view. A reduction to a normal form represented by a closed surface with a finite number of contours is given. The application of this reduction to the question of topological invariants of a singularity of a vector field in three-space is considered.

5. Professor C. L. E. Moore: *Note on surfaces in a non-Riemannian space.*

This paper discusses some of the properties of surfaces in a general space and especially of those whose coordinates satisfy a linear contravariant differential equation.

6. Professor Arnold Emch: *On triple systems and their connection with certain problems of algebraic geometry and analysis situs.*

Since the time when Steiner first drew attention to the possibility of so-called triple systems, combinatory and group properties of such systems have been investigated rather extensively by a number of mathematicians. The purpose of this paper is to show how the geometric aspect of the problem reveals an interesting connection between systems of pairs of triple systems without a common triad and a class of algebraic surfaces passing singly or multiply through each of the $(1/2)n(n-1)$ joins of n generic points in euclidean or projective 3-space. It is further shown that from the standpoint of analysis situs two such triple systems are equivalent with either one- or two-sided complexes of 2-cells of a certain genus, which in every case may easily be determined. As such complexes are triangulated from the start, the problem of their mapping on plane polygons offers no particular difficulty, as is shown by a number of examples.

7. Professor Virgil Snyder: *On a problem in closure.*

This paper will appear in full in an early issue of this BULLETIN.

8. Professor N. Altshiller-Court: *On the de Longchamps circle of the triangle.*

C. de Longchamps considered the circle, now bearing his name, orthogonal to the three circles (A, a), (B, b), (C, c), described from the vertices of a triangle with radii equal to the opposite sides (JOURNAL DE MATHÉMATIQUES SPÉCIALES, 1886, pp. 57 etc.). The present paper considers mainly the circles of similitude and the bisecting circles of these circles taken in pairs. The circles of similitude are the symmetrics of the circles of Apollonius of the basic triangle with respect to the perpendicular bisectors of the corresponding sides. If the angles of the triangle are acute, the three circles of similitude have two real points in common, each of which has its distances from the vertices of the triangle proportional to the opposite sides; and these two points are the only points of the plane having this property. They lie on the Euler line of the triangle and are the limiting points of the coaxial system of circles determined by the de Longchamps circle, the circumcircle, and the nine-point circle of the basic triangle.

9. Professor T. Y. Thomas and Dr. A. D. Michal: *Concerning differential invariants in certain new differential geometries.* Preliminary communication.

The authors study differential invariants in connection with the theory of affinely connected manifolds of the geometry of paths and with the theory of relative quadratic differential forms, by means of the differential equations defining the invariants. Infinitesimal transformations of tensors and r-parameter group considerations enter in the discussion. An attack is made on equivalence problems by means of a new concept introduced in connection with the determination of the number of independent invariants. It is intended to apply and extend the methods to the treatment of projective differential invariants.

10. Mr. J. H. Neelley: *The invariant conditions for some compound singularities of the plane rational quartic.*

Using invariants, as given by R. M. Winger in the AMERICAN JOURNAL OF MATHEMATICS, vol. 36, this paper develops invariant conditions for the plane rational quartic curve to have a cusp of the second kind, sometimes called a ramphoid cusp. The invariant condition for the compound singularity due to the coincidence of three nodes in a manner to form an oscnode have also been obtained by a very neat method of representation which is unique to this type of curve.

11. Professor J. W. Young: *On a form of general geometry.* Preliminary communication.

In a paper on the geometries associated with a certain system of Cremona groups, TRANSACTIONS OF THIS SOCIETY, vol. 17, p. 233, the author, in collaboration with F. M. Morgan, showed that a certain system of Cremona groups in a space S_n, $n=1, 2, 3, \cdots$, could be represented by means of a linear associative algebra on the points of S_n in the form $x'=(ax+b)/(cx+d)$. This representation enabled them to derive certain fundamental theorems of the geometries associated with these groups by using elementary properties of the projective group on a line. The present paper derives similar theorems in a much more general setting. Given a class S of elements $a, b, c, \cdots$ (which may be points, or lines, etc., but need not be geometric) and a linear associative algebra A on the elements of S (a previous paper, not yet published, deals with the construction of such algebras). If multiplication is commutative in A, theorems, analogous to those referred to above, are readily derived for the geometry associated with the linear fractional group on a single variable in A. The paper also makes a beginning toward extending its methods to algebras on S in which multiplication is not commutative.

12. Professor J. A. Shohat: *On certain applications of Tchebycheff polynomials in several variables.*

This paper solves the following problem: Given an arbitrary polynomial $G_n(x_1, \cdots, x_s)$ of degree $\leqq n$, in s independent variables, find the maximum of the absolute value of an arbitrary given homogeneous polynomial of degree $m(=1, 2, 3, \cdots)$ of the coefficients of G_n if we know the value of $\int_D G_n^2(x_1, \cdots, x_s)\, q(x_1, \cdots, x_s)dx_1 \cdots dx_s$, or of max. $|q(x_1, \cdots, x_s)\, G_n(x_1, \cdots, x_s)|$ in D, where D is a certain s-dimensional domain, finite or infinite, and q is defined in D. The method used here is an extension of that given by the author in a paper presented to the Society in September, 1925. We utilize Tchebycheff polynomials in several variables. Numerous applications to Tchebycheff polynomials and to polynomials in general are obtained.

13. Professor H. L. Olson: *Linear congruences in quaternion arithmetic.*

This paper extends to the arithmetic of rational integral quaternions the concept of a linear congruence. It presents a general process of solution and a criterion for the solvability of a linear quaternionic congruence.

14. Professor H. L. Olson: *Linear congruences in a linear algebra.*

This paper presents the condition for the existence of a solution of a linear congruence $axb \underset{l}{\equiv} c$ (mod m) or $axb \underset{r}{\equiv} c$ (mod m) in any rational semi-simple linear algebra of finite order. In the case in which the norms of a and b are relatively prime to the norm of m, the solution is exhibited in explicit form.

15. Professor G. A. Miller: *Subgroups of index p^2 contained in a group of order p^m.*

The author proves that the number of the invariant subgroups contained in a group G of order p^m which correspond to cyclic quotient groups of order p^2 is of the form $p^{\alpha-1}(1+p+p^2+\cdots)$ where α is the number of the independent generators of G, whenever there is at least one such invariant subgroup in G; also, that if a group of order p^m contains non-invariant subgroups of index p^2, then every set of such conjugate subgroups generates a subgroup of index p which involves at least one set of p conjugate subgroups of index p^2 whose common cross-cut is of order p^{m-3} and is invariant under the entire group.

16. Professor G. A. Miller: *The so-called Naperian logarithms.*

This paper calls attention to the fact that the numbers which Napier called logarithms do not satisfy the fundamental law on the logarithm of the product of two numbers. It is also noted that the equation Nap.

$\log y = 10^7 \log_e (10^7/y)$ which is commonly found in our histories of mathematics is inexact. In his well known lecture on *John Napier and the invention of logarithms*, 1614, E. W. Hobson refers to this equation as expressing "the exact relation" if Nap. log y represents the logarithm of y in accordance with Napier's tables. It is observed that this conflicts with Lindemann's theorem that in the equation $y = e^x$, x and y are both algebraic only when $x = 0$. The author concludes that it is undesirable to call logarithms to the base e Naperian logarithms. The base e was first used as a definite number long after Napier and Speidell published their tables. Statements relating to the development of logarithms are frequently obscured by a lack of distinction between exact relations and approximate ones.

17. Dr. H. A. Bender: *Concerning an important theorem in group theory.*

The theorem concerning simply isomorphic groups is one of the most important theorems in group theory, yet with the present development of the theory of groups it is practically useless for determining all the distinct groups of a given class. This is shown by several examples; illustrations are also given of the misuse of this theorem. Suggestions are given for the development of certain phases of the theory of groups which will aid materially in the application of the above theorem.

18. Professor O. E. Glenn: *A symbolical modular theory of biternary concomitants.*

The third lemma in the group of theorems preliminary to Gordan's third proof of Gordan's theorem finds its analog, in this paper, for the case of biternary transvectants $(f, g)_{kl}^{ij}$, as follows: If a finite system (A) of connexes, $A_1, A_2, \cdots$, all of which are concomitants of $f = a_x^m \alpha_u^n$, includes f and is relatively complete for a set (G) of constant symbolical moduli $G_1, G_2, \cdots$, and if a finite system (B) of connexes $B_1, B_2, \cdots$, which are also concomitants of f, is also relatively complete for a set (Γ) of moduli $\Gamma_1, \Gamma_2, \cdots$ and contains one form for each modulus $G_1, G_2, \cdots$ whose only symbolic factor is precisely that modulus, then the system (C) derived by transvection between (A) and (B) is relatively finite and complete modulo (Γ). The moduli are of the various forms $a_\alpha^r a_\beta^s \cdots b_\gamma^t \cdots, a_\alpha^q a_\beta^r \cdots b_\gamma^s (abc)^t, (abc)^r, \cdots$. With this lemma a succession of theorems furnishing methods for the generation of fundamental systems of connexes is completed. The known system of $a_x \alpha_u$ (Clebsch and Gordan) is now found with surprising brevity; the extensive system of $a_x^2 \alpha_u$ is obtained by very direct processes. Further investigation is being made concerning the uses of Gordan's lemma and its extensions, in the symbolic modular theory of biternary forms. The generalization has been made by Wood, from the known binary theory, to the case of covariantive ternary perpetuants.

19. Professor C. C. MacDuffee: *Ideals in linear associative algebras.*

This paper is an introduction to the study of ideals in linear associative algebras, using Dickson's definition of set of integral elements. It is shown that if every left ideal is principal, every pair of integral numbers has a greatest common right divisor, and conversely. A greatest common right divisor is unique up to a unit left factor if its norm is not zero. The corresponding theorems hold for greatest common left divisors. The property of unique factorization in a non-commutative algebra does not always follow from the existence of the greatest common divisors. Dedekind's ideals are of little use in this problem, since $\alpha = \beta\gamma$ does not always imply $\{\alpha\} = \{\beta\}\{\gamma\}$. This difficulty is avoided by introducing ideals of another sort, namely, by defining $\{\alpha\}$ to be the first matrix R_α of α. Since matrices with integral elements are factorable into primes in a manner which is unique except for unit factors, these ideals restore unique factorization to every rational semi-simple algebra.

20. Professor L. E. Dickson: *All integral solutions of* $ax^2+bxy+cy^2=w_1 w_2 \cdots w_n$.

This paper appears in full in the present issue of this BULLETIN.

21. Professor H. S. Vandiver: *A necessary and sufficient condition that an integer be prime.*

The theory of definite quadratic forms is used to establish the following theorem: Consider all the sets of integers (x, y) which satisfy $x^2+ay^2=kn$, where a and n are given integers, $a<n$ and not a square, a prime to n; an integer v exists such that $v^2+a\equiv 0 \pmod{n}$; k is an integer not having a square factor different from unity dividing y^2, such that $k \leqq 2(a/3)^{1/2}$, and x and y are positive integers or zero. Then the integer n is prime if and only if there is a unique set (x', y') satisfying the above conditions and such that x' and y' are each prime to n. The criterion may be employed with advantage when n is large.

22. Professor H. S. Vandiver: *On the theory of finite rings.*

In a paper in the TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), p. 293, the writer considered a theory of finite algebras, defined as systems consisting of a finite number of elements which combine according to the laws of elementary algebra, except that division is not always possible and when possible is not always unique. In accordance with present day terminology such a system is called a ring. Kircher showed (AMERICAN JOURNAL, vol. 39 (1917), p. 279) that non-units in a ring did not always decompose uniquely into prime non-unit factors. The present paper investigates this question further. Two elements whose difference is a unit are called *absolutely distinct.* If, in comparing two factorizations of a non-unit element into prime non-unit factors we regard as distinct only those elements which are absolutely distinct, the writer shows that for an extensive class of rings this type of decomposition is unique, and conjectures that the theorem is general.

23. Professors H. S. Vandiver and G. E. Wahlin: *Report on certain topics in the theory of algebraic numbers.*

The aim of this report is to supplement the report on the theory of algebraic numbers of the Committee on Algebraic Numbers of the National Research Council published in the BULLETIN OF THE NATIONAL RESEARCH COUNCIL, Feb. 1923. The class number in any algebraic field, irregular cyclotomic fields and the application of cyclotomic fields to Fermat's last theorem, are the topics covered in the present report; the development of each of these subjects is traced from the beginning of its history. The report will be published by the National Research Council.

24. Professor O. C. Hazlett: *Formal modular protomorphs.*

This paper proves the principal theorems of a recent paper by W. L. G. Williams, TRANSACTIONS OF THIS SOCIETY, vol. 28 (1926), p. 183, under much weaker hypotheses than those used by him. He proved that if $GF[p^n]$ is any Galois field of order p^n, and if $f(a;x)$ is any binary form for which no one of the binomial coefficients is congruent to zero, modulo p, then a complete set of formal modular protomorphs of f with respect to $GF[p^n]$ consists of a complete set of algebraic protomorphs of f together with $a_0 a_1^{p^n} - a_0^{p^n} a_1$. Similarly for a system of forms. The present paper proves these theorems under the hypothesis that f is such that its order is not congruent to 0 modulo p.

25. Professor J. W. Young: *On the partitions of a group.*

A partition of any group G is a class $[H]$ of subgroups of G such that every element $(\neq 1)$ of G is in just one H. Let $G_{1,s}$ be the group generated by any element $s(\neq 1)$ of G, and let $G_{k+1,s}$ be the group generated by all the $G_{k,t}$ containing $s(k=1, 2, 3, \cdots)$. If there exists a smallest n (finite or transfinite) such that for every s of G, $G_{n,s}=G_{n+1,s}$, G is said to be of type n. If G is of type n the class of distinct subgroups $[G_{n,s}]$ as s ranges over G forms a partition, and conversely. The partition obtained in this way is called primitive. A partition consisting of two or more distinct subgroups is called proper. Any abelian group is of type $\leqq 4$; if it contains no elements of finite order, of type $\leqq 2$. The primitive partition of an abelian group G is proper, if and only if, either (1) G contains no elements of finite order, or (2) all the elements of G are of the same prime order. A partition H of a group G is said to be an L-partition if the group K generated by any finite set $H_1, H_2, \cdots, H_k$ of the partition is such that any other H_l either is entirely contained in K or has no element $(\neq 1)$ in common with K. Other theorems relate to the conditions on G under which the primitive partition of a group is an L-partition.

26. Professor J. W. Young: *A new formulation for general algebra.* Second paper.

This paper gives a simpler and more symmetric set of properties on which to base the formulation given in a previous paper with the same title (this BULLETIN, vol. 19, pp. 514, 515).

27. Professor M. H. Ingraham: *The infinite case of the equivalence of linear associative algebras to matrix algebras.* Preliminary report.

If an algebra A over a field F has a finitely linearly independent base in terms of which all its elements may be expressed as finite linear combinations with coefficients in F, then A is equivalent to an algebra of matrices, infinite in order if A is infinite in order, with elements in the field and such that each column contains only a finite number of non-zero elements. A is reciprocal to an algebra of matrices of the same type. Such a base exists if A has a well-ordered base, not necessarily linearly independent. Certain classifications of elements and sub-algebras are given. The problem of the extension of the notion of the minimum equation of an element and the definition of integral elements is discussed.

28. Mr. M. E. Mullings: *The rotational derivative and some applications.*

This paper gives a definition of a rotational derivative of a vector function analogous to the directional derivative of a vector or scalar function (see Gans, *Einführung in die Vektoranalysis*, Leipzig, 1921, p. 34). For a simple surface and a curve of type C, two important auxiliary theorems are obtained. By use of his definition, the Duhamel-Moore theorem, and the two auxiliary theorems, the author gives a completely satisfactory proof of Stokes' theorem which is simpler and more direct than previous proofs.

29. Dr. C. C. Camp: *An expansion connected with a partial differential equation.* Preliminary communication.

There will be suitable characteristic solutions of $\partial u/\partial x+\partial u/\partial y+(\lambda f+g)u=0$ and the boundary conditions $u(x, \pi)=u(x, -\pi)$, $u(\pi, y)=u(-\pi,y)$, if f and g are each composed of terms in x and y alone, besides one of the form $DA(x, y)\equiv(\partial/\partial x+\partial/\partial y)A(x, y)$, provided A also satisfies the boundary conditions. Such a function in f or g does not influence the values of λ. By the transformation $u=u_1e^{-\lambda A-B}$, the equation may be reduced to a simpler form. By writing $u=X(x)Y(y)$, this can then be broken up into a system of ordinary equations which are new because of the way in which the parameters λ, μ enter. By proper choice of new parameters, the author builds up the analog of the Green's function; he investigates the convergence of the expansion of an arbitrary function $F(x, y)$ by using an extension of the contour integral method of Birkhoff.

This treatment seems to exhaust the possibilities for the given system although the theory may be extended to n variables and to less simple boundary conditions.

30. Professor O. D. Kellogg: *On some theorems of Bôcher concerning isolated singularities of harmonic functions.*

This paper appears in full in the present number of this BULLETIN.

31. Mr. L. B. Robinson: *On equations in mixed differences; continuation.*

In this paper (see this BULLETIN, vol. 32, p. 316) the author considers the equation $u'(x)=\sum_{i=1}^{n}[P_i(x)/\{x(x+r)\}]u[a_ix+\delta_i]+Q(x)/\{x(x+r)\}$, and shows that if three inequalities are satisfied a single infinity of solutions exists, each with a logarithmic singularity at $x=0$ and $x+r=0$.

32. Professor P. R. Rider: *A figuratrix for double integrals.* Second paper.

In the first paper by this title the author defined a figuratrix for double integrals in the calculus of variations and showed that a certain necessary condition obtained by Kobb implied that the total curvature of the figuratrix must be positive. In this paper he gives geometric interpretations of the transversality condition and the E-function.

33. Professor C. A. Garabedian: *Disk of constant or variable thickness in uniform or accelerated rotation.*

By applying to the problem of the rotating disk the method of series elaborated in the author's *Circular plates of constant or variable thickness* (TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), p. 343), it is possible to bring under a uniform treatment some aspects of the subject hitherto studied separately. When the thickness is constant, the results for uniform rotation include certain solutions due to Chree, while the results for accelerated rotation show that the formulas recently derived by Prudon and by Pigeaud on the assumption of a thin disk are equally applicable to the thick disk. In the case of a disk of variable thickness in uniform rotation, it turns out that the leading term in the solution in series is given by the Stodola differential equation; subsequent terms in the series offer a means of judging the accuracy of the Stodola approximation. Similar results are obtained for the disk of variable thickness in accelerated rotation, and here the differential equation analogous to that of Stodola appears to be new. An abstract will be found in the COMPTES RENDUS, vol. 183, Aug. 17, 1926, p. 416.

34. Professor F. D. Murnaghan: *The Boltzmann-Hopkinson principle of superposition in Maxwell's theory of the layer dielectric.*

In this paper, which is a continuation of one presented to the Society at the New York meeting, May 1, 1926, the author gives a proof of the empirical principle of superposition by which the electric intensities in a layer dielectric for any applied E.M.F. may be calculated when the intensities corresponding to a constant E.M.F. are known. The principle is sometimes stated as valid for the current but this is erroneous. However, the expression follows at once from the expressions for the various electric intensities.

35. Professor R. L. Wilder: *On the definition of simple closed curve.* Second paper.

In an earlier paper of the same title presented to this Society on January 1, 1926, the author gave a definition of a simple closed curve M which required neither that M be bounded nor that it be closed. However, this definition did require that certain subsets of M should be closed in M. The following two definitions do not subject M to any of the above restrictions: (1) A simple closed curve is a connected and regular point set which remains connected upon the omission of any connected subset; (2) A simple closed curve is a connected and regular point set which contains no cut-points and is disconnected by the omission of any two of its points.

36. Dr. H. M. Gehman: *Concerning irreducibly connected sets and irreducible continua.*

Every bounded continuum is irreducibly connected about the set consisting of its non-cut-points. If M is any connected set, which is irreducibly connected about one of its subsets B, then B contains all the non-cut-points of M. Therefore, a necessary and sufficient condition that a bounded continuum M be irreducibly connected about one of its subsets B, is that B contain all the non-cut-points of M. Every bounded continuum is an irreducible continuum about the set consisting of its non-cut-points. If M is any continuum which is an irreducible continuum about a subset consisting of n points, but not about any subset consisting of $n-1$ points, and if A is any closed subset of M such that M is an irreducible continuum about A but not about any proper closed subset of A, then A consists of exactly n points.

37. Dr. H. M. Gehman: *Concerning end points of continuous curves and other continua.*

The property used by K. Menger (MATHEMATISCHE ANNALEN, vol. 95 (1925), p. 277) to define an end point is equivalent in the case of a continuous curve, to the property used by R. L. Wilder (FUNDAMENTA MATHEMATICAE, vol. 7 (1925), p. 358). In the case of a bounded continuum, Menger's property is stronger than Yoneyama's property A, and independent of his properties B and C (TÔHOKU MATHEMATICAL JOURNAL, vol. 18 (1920), p. 253). It is shown that if a point P of a bounded continuum M either has Menger's property, or has Yoneyama's properties

A and C, then M is connected im kleinen at P, and if P is on the boundary of a domain complementary to M, then P is accessible from that domain. Examples are given to show that this theorem does not remain true if the above properties are replaced by certain other weaker properties

38. Mr. G. T. Whyburn: *The most general closed point set over which a continuous function may be defined by certain properties.*

This paper will appear in full in an early issue of this BULLETIN.

39. Mr. G. T. Whyburn: *Concerning certain types of continuous curves.*

This paper establishes the following results: (1) A bounded continuum M is a continuous curve if and only if every connected open subset of M has property S (see R. L. Moore, FUNDAMENTA MATHEMATICAE, vol. 3 (1921), p. 232). (2) A bounded continuum M is an acyclic continuous curve if and only if every connected subset of M is uniformly connected im kleinen. (3) A bounded continuum M is a continuous curve if and only if the set of all non-end points of M is uniformly connected im kleinen. (4) A bounded continuum M is a continuous curve if and only if the set of all non-end points of M has property S. (5) Every strongly connected subset of a continuous curve every subcontinuum of which is a continuous curve is strongly connected im kleinen. (6) The set of all non-end points of a continuous curve M is a subset of the sum of the boundaries of the complementary domains of M provided that if A and B are any two points belonging to the same simple closed curve of M, then $M-(A+B)$ is not connected.

40. Dr. B. O. Koopman: *Note on the extension of a method of Briot and Bouquet for the reduction of singular points.*

This paper appears in full in the present issue of this BULLETIN.

41. Mr. J. H. Sturdivant: *Properties of second-order linear systems with bounded, Lebesgue integrable coefficients.*

Bôcher (*Leçons sur les Méthodes de Sturm*, Paris, 1917) and Ettlinger (TRANSACTIONS OF THIS SOCIETY, vols. 19 and 22) have demonstrated properties of the zeros of a solution of second-order linear systems with coefficients continuous in x and the parameter as independent variables, and of a linear combination of a solution with its first derivative, for both the Sturmian and the general self-adjoint types involving values at two points. By making use of well known properties of Lebesgue integrals and of a forthcoming, extended implicit function theorem due to Ettlinger, this paper extends the above results to systems with coefficients continuous in the parameter, but only bounded and Lebesgue integrable in x.

42. Mr. W. A. Rees: *The method of successive approximations for a second order ordinary linear differential equation with bounded measurable coefficients.*

The method of successive approximation, developed by Peano, Picard and Bôcher, is here applied to ordinary linear differential equations of the second order with coefficients that are bounded and measurable on $a \leqq x \leqq b$. The method follows that of Bôcher (*Leçons sur les Méthodes de Sturm*, Paris, 1917, Chapter I). In addition the existence theorem is established for the system of two ordinary linear differential equations of the first order, $y' = K(x)z + J(x)$; $z' = G(x)y + H(x)$; $y(a) = \alpha$, $z(a) = \beta$, where K, G, H, and J are bounded measurable functions of the real variable x on $a \leqq x \leqq b$, and α and β are real constants. This system is more general than a system of the second order but reduces to the latter if $K(x)$ is continuous and has a bounded measurable derivative $K'(x)$ at every point on $a \leqq x \leqq b$ except for possibly a null set. For this particular case, the existence theorem may be regarded as a second method of obtaining a solution of an ordinary second order system. The errors after n approximations are evaluated and compared for the two methods.

43. Professor H. J. Ettlinger: *Linear boundary value problems with summable coefficients involving integral auxiliary conditions.*

This paper deals with a system of n ordinary linear differential equations of the first order, whose coefficients are summable functions. By means of the adjoint system a necessary and sufficient condition is obtained for the equivalence between the most general type of linear auxiliary integral condition and a linear boundary condition of the usual type involving the values of the unknown functions at the end points. The above result is carried over to an ordinary linear differential equation of the nth order. One of the results is that the self-adjoint pair of equations $y' = K(x)z$, $z' = G(x)y$, where $K(x)$ and $G(x)$ are summable functions of x, with the integral auxiliary conditions $\int_a^b r_i(t)y(t)dt = 0$, $i = 1, 2$, where r_1 and r_2 are independent summable functions, can be carried over into the most general self-adjoint system of the second order involving values of y and z at two points (see TRANSACTIONS OF THIS SOCIETY, vol. 19, p. 79; vol. 22, p. 136).

44. Professor H. J. Ettlinger: *Existence theorems for implicit functions of real variables.*

The classical theorems on implicit functions make use of the continuity of the given functions and their partial derivatives when all the variables are considered as independent. The author wishes to establish the existence of unique and continuous solutions of a system of equations under conditions which demand a great deal less than the above. For a function of the two variables the following theorem is proved: If $F(x, y)$ is defined in the neighborhood of (X, Y); $F(X, Y)=0$; $F(x, Y)$ is continuous in x; $F_y'(x, y)$ exists for every x and for every y almost everywhere, and is bounded; F_y' is measurable in y for every x and continuous in x for every y almost everywhere; $F_y' \geqq 0$ (or $\leqq 0$), the equality sign holding for any x on a null set only, then for some neighborhood of (X, Y) there exists a single-valued continuous function $y=y(x)$ such that $Y=y(X)$ and $F(x, y(x)) \equiv 0$. Additional hypotheses concerning the existence of F_x with properties symmetric to those of F_y', the continuity of F_y' in y for fixed x, the continuity of F_x' in x for fixed y, insure the existence of the derivative $y'(x)$, wherever $F_y' \neq 0$. Corresponding theorems are given for a system of n functions in n dependent variables and m independent variables.

45. Professor H. J. Ettlinger: *R. L. Moore's principle and its converse.*

R. L. Moore (Annals of Mathematics, (2), vol. 13, p. 161) has stated an important principle which the author has made use of in various connections. It is the purpose of this paper to restate this theorem in terms of a sequence of *horizontal functions* defined on a point set E in an m-dimensional closed domain. More general sufficient conditions are obtained which insure the validity of the principle. Necessary conditions are also obtained. Finally, necessary and sufficient conditions are found. The generalization of Moore's principle contained in this paper corresponds to the passage from the bounded measurable case of integrability to the general summable case. Applications are made to obtain properties of summable functions.

46. Professor H. J. Ettlinger: *On the inversion of the order of integration of a two-fold iterated integral.*

Based on the work of W. H. Young, Lichtenstein, and Gillespie, a necessary and sufficient condition is obtained for inverting the order of a two-fold iterated integral for which the double integral does not necessarily exist. More general sufficient conditions than those given by the above writers are also obtained. From the principal theorems, several corollaries follow relative to integrable functions of a parameter defined by a definite integral.

47. Professor H. J. Ettlinger: *The kinetics of learning.*

The laws of certain specified types of learning process, in particular, a mechanical process, which may be roughly represented by learning to operate a typewriter, may be embodied in a first order linear differential (or difference) equation with constant coefficients. From this relation follows a set of equations which express chiefly by means of exponential functions (or binomial expressions) the quantitative aspects of the learning process and exhibit it as a form of "growth." It is not unlikely that equations of this kind may prove useful in analyzing personnel performance, in factory and offices, as well as predicting future attainment.

48. Dr. M. H. Stone: *Integrals analogous to Fourier integrals.*

In this paper the problem of representing an arbitrary function in terms of integrals which generalize the Fourier integral as Sturm-Liouville series do Fourier series, is considered. The basis of the treatment is a passage from a regular differential system of the second order on an interval (a, b) to the limiting cases $a=0, b=+\infty$, and $a=-\infty, b=+\infty$. Hankel's integral in terms of Bessel functions is also discussed by means of formulas suggested by the preceding work. The results may be described roughly by the statement that these integrals are equivalent to the Fourier integral on any interval completely interior to the infinite interval of definition, with respect to the class of all functions absolutely summable on that infinite interval.

49. Professor Marston Morse: *A theory of periodic extremals "in the large."*

In applying the calculus of variations to the study of differential equations, most frequently it is not a minimum or maximum that is sought. Under very general conditions the search for extremals joining two fixed points or deformable into a given closed curve can be reduced to that for the critical points of a function f of a large but fixed number of variables. Of the $n+1$ types of critical points only the two extreme types can give maxima or minima. The nature of the critical point of f corresponding to a closed extremal g is shown to depend in an exact manner upon the number of conjugate points on g, the number of periodic, linearly independent solutions of the corresponding Jacobi differential equation, and upon the angle at which a neighboring extremal returns to intersect itself, if it returns at all. All this is "in the small." The basis for the results "in the large" are the author's *Relations between critical points*, TRANSACTIONS OF THIS SOCIETY, vol. 27 (1925), p. 345.

50. Dr. A. D. Michal: *A generalization of Cartan's integral invariants.*

This paper considers non-additive functional invariants which generalize, and include as a special case, Cartan's integral invariants. The author points out that Cartan's process of stepping from an invariant in Poincaré's sense to a complete Cartan invariant can be extended to certain non-additive functional invariants when the functional invariants in Poincaré's sense are of a particular type. Properties of tensors whose components depend on i points are found useful.

51. Dr. L. S. Hill: *Transformation properties of certain general regions and of their frontiers.*

A general theorem is established which has as a corollary the important theorem that a homeomorphism between two simple closed Jordan curves in the euclidean plane may be extended to the bounded regions determined by the curves. A bounded simply-connected region R in the euclidean plane is considered, whose frontier F is the sum of two "separable" continua irreducible between two points and is also the frontier of the unbounded component of its complement. It is shown that such a frontier F may be represented as the "supracontinuous" image of the circumference C of a circle; and that the supracontinuous function thus defined over the circumference C may be extended to have the range $C+D$, where D is the interior of the circle, in such manner that the extended function is supracontinuous over $C+D$, and images the set $C+D$ on the set $F+R$. It is further shown that the correspondence thus established between the region R and the interior D of the circle is a homeomorphism.

52. Dr. L. S. Hill: *Properties of certain aggregate functions.*

In Hahn's recent work on the "prime parts" of a continuum, and in papers by others, notably R. L. Moore, the idea has been suggested of treating aggregates of continua as functions with properties of upper semicontinuity, etc. In the present paper, a function $X_t=f(t)$ is studied, the range $T=\{t\}$ of the variable being a set of points in a euclidean space, and the function "value" X_t a set of points in the same, or another, euclidean space. Under suitable restrictions, the properties of such functions are investigated and exhibited in convenient form. Generalizations of theorems in classical real analysis are incidentally obtained. It is shown, for example, that if $T=\{t\}$ is either a region or a perfect set, and the function $X_t=f(t)$ is upper semicontinuous, or lower semicontinuous, at each point of T, then the function is continuous at each point of a set which is everywhere dense in T.

53. Professor I. M. Vinogradov: *On a general theorem concerning the distribution of residues and non-residues of powers.*

In the present paper the author offers a new method for solving certain questions regarding the distribution of residues and non-residues of powers, discussed in his earlier papers of 1916–18. The difference between this method and that previously used lies in its entirely elementary character.

54. Professor W. M. Whyburn: *On a difference system of the second order connected with boundary value problems for differential systems.*

The author treats the system of difference equations

$$z_{in}(\lambda)=K_{in}(\lambda)\cdot\Delta y_{in}(\lambda)/\Delta n,\quad \Delta z_{in}(\lambda)/\Delta n=G_{in}(\lambda)y_{i+1,n}(\lambda),$$

defined over the finite point set $E_n=\{x_{in}\}$ lying on an interval $a\leqq x_i\leqq b$; K_{in} and G_{in} are continuous functions of λ on $L_1<\lambda<L_2$ but their derivatives with respect to λ need not exist. He discusses (1) the separation of the nodes of two linear combinations of the type $\Phi_{in}=\alpha_{in}z_{in}-\beta_{in}y_{in}$ for any fixed value of λ; (2) conditions for non-oscillation of Y and Φ; (3) necessary and sufficient conditions that the zeros of Y move to the left as λ increases; (4) existence and oscillation theorems for Sturmian boundary conditions; (5) the Green's function. He shows, furthermore, that by

means of a sequence of horizontal functions having a measurable function as the limit function almost everywhere on $a \leqq x \leqq b$, the above equations and many of the above results go over as $n \rightarrow \infty$ to a differential system defined over a limiting set E, in which the coefficients are measurable functions (see forthcoming papers by the author and by J. H. Sturdivant). Certain of the above results are established for the more general system $\Delta y_{in}/\Delta_n = A_{in}z_{in} + B_{in}y_{in}$, $\Delta z_{in}/\Delta_n = D_{in}z_{in} + E_{in}y_{in}$.

55. Dr. J. M. Thomas: *Note on a differential equation.*

The differential equation $aX' + bX + cY' + dY = 0$, where a, b, c, and d are given functions of the independent variables x and y, X and Y are unknown functions of x and y respectively, and X' and Y' their derivatives, is shown to be equivalent to a system of total differential equations. The equations can therefore be treated by familiar methods.

56. Professor Norman Miller: *On related maxima and minima.*

This paper is concerned with the character of the extremes of n functions of n real variables at a minimizing point, each function being minimized under the condition that the others retain the constant values which they have at that point. Application is made to the elementary symmetric functions and certain sufficient conditions are developed for determining the character of the several extremes of these functions in the case that two of the n variables have equal values.

57. Professor W. Sierpinski: *Nuclear points in the theory of abstract sets.*

This paper appears in full in the present number of this BULLETIN.

58. Professor V. W. Niemytski: *On the third axiom of metric space.*

This article is devoted to the proof that any space which satisfies the following conditions is metrizable. The points of accumulation of the space may be defined in terms of a symmetric, non-negative distance function $\delta(x, y)$ which vanishes if and only if $x = y$, and has the property that if $L\delta(x, y_n) = 0$, $L\delta(y_n, z_n) = 0$, then $L\delta(x, z_n) = 0$. Two proofs are given, one based on an article by A. D. Pitcher and E. W. Chittenden (TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 66), the other on the necessary and sufficient conditions for the metrization of a class (L) formulated by Alexandroff and Urysohn (COMPTES RENDUS, vol. 177, p. 1274).

59. Mr. G. P. Aldrich: *On the complete existential theory of ten properties of the classes (V) of Fréchet.*

This paper presents the results of an investigation of the interrelations of ten fundamental properties of families of neighborhoods. Certain combinations of these properties are found to be completely independent.

Of the 1024 possible combinations of the ten properties and their negatives it is found that only 150 correspond to existing spaces. The 150 examples required to complete the discussion are given. The paper contains the following appendices: On the complete independence of the axioms of Hausdorff for topological space; a postulational definition of betweenness in terms of neighborhood; a definition of point of accumulation in terms of dyadic relations.

60. Miss Fay Farnum: *On triadic Cremona nets.*

A birational correspondence between the points of two planes is called triadic if, in each of the two planes, there are only three groups of fundamental points. Examples of this case have been published by Ruffini and by Montesano. The present paper intends to exhibit further cases of this kind. The author develops (a) triadic correspondences, as products of previously existing solutions; (b) triadic correspondences having simple basis points in both planes; (c) some triadic correspondences having simple basis points in one plane only.

61. Professor J. W. Campbell: *A periodic solution for a certain problem in mechanics.*

If a mass m resting on a smooth horizontal table is connected by a light inextensible string, which passes through a hole in the center of the table, with a mass M hanging freely, then under certain conditions when m revolves about the hole, M will execute periodic motion in a vertical line. A power series solution is obtained for this periodic motion of M when the amplitude is not too large.

62. Professor E. P. Lane: *The contact of a cubic surface with an analytic surface.*

Using Wilczynski's canonical form of the differential equations of a surface referred to its asymptotics, the author investigates the nature of a contact of a cubic surface with an analytic surface, and obtains necessary and sufficient conditions that a surface may be a cubic. The differential equations being written in the form $y_{uu}+2by_v+fy=0$, $y_{vv}+2a'y_u+gy=0$, the conditions that an integrating surface may be a cubic may be reduced to $a'=b, f=(3/2)b_v+(1/4)(b_{uu}/b)-(5/16)(b_u/b)^2$, $g=(3/2)b_u+(1/4)(b_{vv}/b)-(5/16)(b_v/b)^2$.

63. Professor H. J. Ettlinger: *On the zeros of functions associated with a linear system of the second order.*

Two lemmas are stated concerning the zeros of continuous functions on an interval: If $f(x)$ is a real continuous function of x on I, $a\leqq x\leqq b$, the set of zeros of $f(x)$ is closed; if $f(x)$ has a derivative at every point of I and if $f^2+f'^2>0$ on I, the set of zeros of $f(x)$ on I is finite in number. By means of these lemmas, the author establishes theorems concerning the number and separation of the zeros of the functions $y(x)$ and $z(x)$, and linear combinations of these, defined by the differential system $y'=K(x)z(x)$, $z'=G(x)y(x)$, $y(a)=A$, $z(a)=B$ where $K(x)$ is positive and continuous on I and $G(x)$ is bounded and measurable, A and B being real constants not both zero.

64. Miss L. W. Griffiths: *Certain quaternary quadratic forms and diophantine equations.*

A unified treatment of sets of integral elements in the generalized quaternion algebras (see Dickson, *Algebras and their Arithmetics*, p. 187) in which the parameters $(\alpha, -\beta)$ are $(-1, 1)$, $(-2, -3)$, $(-1, -3)$, $(2, -3)$, $(3, -3)$, $(4, -3)$, $(1, 5)$, $(2, 5)$, $(-2, 5)$, $(-3, 5)$, $(1, -7)$, $(-1, -7)$, $(1, 9)$, $(1, -11)$, $(-1, 13)$ yields in application important results concerning the representation of integers in the forms $\sigma^2-\alpha\xi^2+\epsilon\eta^2-\alpha\epsilon\zeta^2+\sigma\eta-\alpha\xi\zeta$, $\sigma^2-\alpha\xi^2-\tau\eta^2+\alpha\tau\zeta^2$, where σ, ξ, η, ζ are integers and $\epsilon=(1-\tau)/4$. The number of such representations is obtained, as are also all solutions of the equations in $k+4$ variables, $k>0$, $\sigma^2-\alpha\xi^2+\epsilon\eta^2-\alpha\epsilon\zeta^2+\sigma\eta-\alpha\xi\zeta=\delta_1\cdots\delta_k$, $\sigma^2-\alpha\xi^2-\tau\eta^2+\alpha\tau\zeta^2=\delta_1\cdots\delta_k$. Certain definite exceptions appear with $(\alpha, \tau)=(3, -3)$, $(4, -3)$, $(-1, -7)$. The method is essentially that of Dickson, (AMERICAN JOURNAL OF MATHEMATICS, vol. 46 (1924)).

65. Professor G. C. Evans: *Note on a theorem of Noaillon and the Dirichlet problem for the circle.*

A necessary and sufficient condition that $u(r, \theta)$ be given by the Poisson integral in terms of some summable boundary function $f(\theta)$ has been given by the author, (COMPTES RENDUS, vol. 177 (1923), p. 241), in terms of uniform absolute continuity of the integral $\int u(r, \theta)d\theta$, where $u(r, \theta)$ is harmonic within the circle; also by Noaillon in terms of convergence in the mean to the function $f(\theta)$ (COMPTES RENDUS, vol. 182 (1926), p. 1371). The latter condition may easily be shown, for harmonic functions, to be equivalent to the former, but even in a slightly amended form it does not involve so essentially the properties of harmonic functions as the former. There is still room indeed for a necessary and sufficient condition which will partake less of the general theory of limits of integrals than any of those so far given but will depend more specifically upon the properties of a function harmonic inside the circle.

66. Professor J. L. Walsh: *On the degree of approximation to an analytic function.*

Let C be an arbitrary limited point set in the z-plane, whose complementary set is a simply connected region B (therefore open). Let $u=\phi(z)$ be a function which maps B on the exterior of the unit circle in the u-plane, so that the infinite points correspond to each other. Let C_R $(R>1)$ denote the Jordan curve $|\phi(z)|=R$, that is, the transform in the z-plane of the circle $|u|=R$. A necessary and sufficient condition that an arbitrary function $F(z)$ which is defined on C be regular-analytic on C is that polynomials $V_n(z)$ of degree n, for $n=0, 1, 2, \cdots$, exist so that we have (1): $|F(z)-V_n(z)| \leqq (M/R^n)$ (M, $R>1$, constant), for every z on C. When the polynomials $V_n(z)$ are given, so that (1) holds, $F(z)$ is regular-analytic in the interior of C_R. When $F(z)$ is regular-analytic in the closed interior of $C_{R'}$, then (1) holds, if the polynomials are suitably chosen, for $R=R'$.

67. Professor Dunham Jackson: *Note on a tensor of the second rank in function space.*

This paper appears in full in the present issue of this BULLETIN.

68. Professor C. E. Wilder: *Reduction of the ordinary linear differential equation of the nth order whose coefficients are certain polynomials in a parameter to a system of n first order equations which are linear in the parameter.*

In this paper the following theorem is proved: The differential equation $(d^n y/dx^n)+P_{n-1}(x, \rho)(d^{n-1}y/dx^{n-1})+\cdots+P_0(x, \rho)=0$, in which $P_k(x, \rho)=\sum_{j=0}^{n-k} P_{n-k,j}(x)\rho^j$, $k=0, 1, 2, \cdots, n-1$, may be reduced to a system of n linear differential equations of the form $(dy_i/dx)=\sum_{j=1}^{n}[\rho a_{ij}(x)+b_{ij}(x)]\, y_j$, $i=1, 2, \cdots, n$, in which the functions $a_{ij}(x)$ and $b_{ij}(x)$ are continuous in a given interval $a \leqq x \leqq b$ if the functions $P_{i+k,i}(x)$ possess $n-k$ continuous derivatives in this interval and if the "characteristic equation" $\sum_{t=0}^{n} P_{n-t,\, n-t}(x)a^t=0$ has roots which are distinct for all values of x in the interval.

69. Professor J. D. Tamarkin: *On Fredholm's integral equations whose kernels are analytic in a parameter.*

This paper is concerned with Fredholm's integral equation whose kernel $K(x, \xi, \lambda)$ is analytic or meromorphic in λ. By using infinite determinants, it is established that the reciprocal is also meromorphic in λ, under certain assumptions as to the kernel $K(x, \xi, \lambda)$. In the special case of the classical Fredholm integral equation a representation is obtained in the form of a quotient of two infinite determinants, which are entire functions in λ, under more general conditions than it has been done before.

70. Professor G. E. Raynor: *On the generalization of Beltrami equations to n-space.*

In the TRANSACTIONS OF THIS SOCIETY for October, 1925, Hedrick and Ingold have obtained a generalization of the Beltrami equations to 3-space. In the present paper the writer obtains equations of the same sort for n-space by a different method and of a somewhat more explicit form.

ARNOLD DRESDEN,
Assistant Secretary.