ISOLATED SINGULAR POINTS OF HARMONIC FUNCTIONS*

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Bôcher has given several theorems relating to the nature of a harmonic function in the neighborhood of an isolated singular point.† E. Picard‡ has proved two of the theorems given by Bôcher in the paper mentioned above. In both papers the following fundamental theorem occurs.

If a function $f(x, y)$ is continuous and harmonic everywhere in the interior of a closed plane region with the exception of an isolated point P in the neighborhood of which $f(x, y)$ tends to plus infinity for every mode of approach to P, f is of the form $c \log (1/r)+V$ in this neighborhood, where c is a positive constant, r the distance from (x, y) to P and V is a function harmonic everywhere in the neighborhood of P, including P itself.

The analogous theorem for three-space is also proved in each paper.

Picard's proof for the plane makes use of complex variables. However, as Bôcher points out, it is desirable to have an independent proof in order to follow out Riemann's idea of basing the theory of complex variables on the theory of real harmonic functions in two variables. Bôcher's discussion is general and applies to harmonic functions in any number of variables. However, both Bôcher and Picard, in the case of the theorem cited for three-space, apply Green's Formula to a region bounded partly by the surface

$$f(x, y, z) = \text{constant}.$$

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* Presented to the Society, October 31, 1925. See also a paper by O. D. Kellogg, *On some theorems of Bôcher concerning isolated singular points of harmonic functions*, which is to appear in the next issue of the BULLETIN.

† This BULLETIN, vol. 9 (1902–3), p. 455 ff.

‡ BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 52 (1924).

Now this involves an integral over this surface and furthermore the integral contains the normal derivative of f on the surface. In order to make the discussion complete it would seem that some consideration of the nature of the surface $f=c$ is necessary in order to know if df/dn has a meaning and if the integral exists. Such a discussion is of course possible but by no means simple. See, for example, the treatment of the analogous problem for Green's Function in the plane in Osgood's *Funktionentheorie*.* It is the purpose of the present paper to obtain for the plane additional results, as well as those of Bôcher, but by a different method. Furthermore, most of the present discussion can be generalized at once to three-space.

We suppose, then, that $f(x, y)$ is a function harmonic everywhere in a plane region A except possibly at a single interior point P. We wish to study the nature of f in the vicinity of P.

About P as a center, and with radius r_1, draw a circle C_1 which lies entirely in the region A. Also about P draw a second circle C_2 with radius $r_2 < r_1$. Let V' be the function which is harmonic everywhere in C_1 and which on C_1 coincides with f. This function we know exists and is given by Poisson's integral. In the following we shall indicate the region bounded by two concentric circles C_i and C_j by the symbol C_iC_j.

Now the difference

$$f - V' \equiv F \tag{1}$$

is continuous and harmonic in C_1C_2 and is zero everywhere on C_1. Let r be the distance of any interior point (x, y) of C_1 to P. Then the function

$$\log \frac{1}{r} - \log \frac{1}{r_1} \equiv W(r)$$

is evidently also harmonic in $C_1 \, C_2$ and takes the value

* First edition, p. 588; 2d ed., p. 673.

zero on C_1. Since V' is given by Poisson's integral, we know that on C_1, dV'/dn exists and is continuous. Also since f is harmonic *on* C_1 its normal derivative also exists and is continuous there. Hence the same is true of the difference $F \equiv f - V'$. The same is seen directly to be true of dW/dn. Also, since both W and F are harmonic *on* C_2 their normal derivatives exist on this circle.

Let us now apply Green's Formula to the region bounded by C_1 and C_2. We have

$$\int_{C_1}\left[F\frac{dW}{dn} - W\frac{dF}{dn}\right]ds = 0,$$

where the normal derivatives are taken toward the interior of the region $C_1\,C_2$. But on C_1, F and W are both zero, and hence

$$\int_{C_1}\left[F\frac{dW}{dn} - W\frac{dF}{dn}\right]ds = 0, \tag{2}$$

or

$$\int_{C_2} F\frac{dW}{dn}\,ds = \int_{C_2} W\frac{dF}{dn}\,ds. \tag{3}$$

But along C_2 we have $dW/dn = -1/r$ which is constant on C_2 as is also W. Hence we have, substituting in (3),

$$\int_{C_2} F ds = -r_2\ W(r_2)\int_{C_2}\frac{dF}{dn}\,ds. \tag{4}$$

But by a well known theorem on harmonic functions

$$\int_{C_1C_2}\frac{dF}{dn}\,ds = \int_{C_1}\frac{dF}{dn}\,ds + \int_{C_1}\frac{dF}{dn}\,ds = 0,$$

and hence (4) can be written

$$\int_{C_2} F ds = r_2 W(r_2)\int_{C_1}\frac{dF}{dn}\,ds. \tag{5}$$

Let us set

$$\frac{1}{2\pi}\int_{C_1}\frac{dF}{dn}\,ds = c.$$

We have then from (5) the result that the function F satisfies the integral equation

$$\int_{C_2} F ds = 2\pi r_2\left(\log\frac{1}{r_2} - \log\frac{1}{r_1}\right) \tag{6}$$

where C_2 is any circle with P as center, and radius $r_2 < r_1$.

Now the general solution of (6) is easily seen to be

$$F = c\left(\log\frac{1}{r_2} - \log\frac{1}{r_1}\right) + \Phi(x,y),$$

where

$$\int_{C_2}\Phi(x,y)ds = 0 \tag{7}$$

and for our purposes Φ must be harmonic in C. Substituting in (1) we have

$$\begin{aligned} f &= c\left(\log\frac{1}{r} - \log\frac{1}{r_1}\right) + \Phi(x,y) + V' \\ &= c\log\frac{1}{r} + \Phi(x,y) + V, \end{aligned} \tag{8}$$

where $V = V' - c\log(1/r_1)$. Since the form of f is of course independent of any circle as C_2 the form (8) holds at any interior point of C_1 except P. Furthermore, on C_1, $\Phi\equiv 0$.

Point P will in general be a singular point of Φ, and we shall show as a matter of fact, unless $\Phi(x, y)\equiv 0$, that in the neighborhood of P the function Φ will tend toward both plus infinity and minus infinity, that is, that there will exist a mode of approach to P along which Φ will tend toward plus infinity and *also* a mode of approach to P for which it will tend to minus infinity. Furthermore, the same statement will be seen to be true concerning the sum

$$c\log\frac{1}{r} + \Phi.$$

To prove this, let us write Φ in the form

$$\Phi = \frac{1}{r}\Psi(x,y). \tag{9}$$

It is clear that in the open region bounded by C_1 and P the function $\Psi(x, y)$ is continuous. We shall now define two functions $f_1(x, y)$ and $f_2(x, y)$ in this open region as follows:

$$\begin{aligned}
f_1(x,y) &= \Psi(x,y) \text{ at points where } \Psi(x,y) \geqq 0,\\
f_1(x,y) &= \quad 0 \quad \text{`` \quad `` \quad ''} \quad \Psi(x,y) < 0,\\
f_2(x,y) &= \Psi(x,y) \text{ `` \quad `` \quad ''} \quad \Psi(x,y) \leqq 0,\\
f_2(x,y) &= \quad 0 \quad \text{`` \quad `` \quad ''} \quad \Psi(x,y) > 0.
\end{aligned}$$

From the continuity of $\Psi(x, y)$ it is clear that f_1 and f_2 are continuous in their region of definition. We have thus

$$\Psi(x,y) = f_1(x,y) + f_2(x,y). \tag{10}$$

From (7), since $1/r$ is constant on any circle with P as center we have

$$\int_C \Psi(x,y)ds = 0,$$

where C is such a circle interior to C_1. Hence

$$\int_C f_1(x,y)ds + \int_C f_2(x,y)ds = 0 \tag{11}$$

or in polar coordinates, with P as pole, and any fixed line through P as polar axis,

$$\int_0^{2\pi} f_1 d\theta + \int_0^{2\pi} f_2 d\theta = 0 \tag{12}$$

for any value of r. Consider now an arbitrary circle C_3 with P as center and radius $r_3 < r_1$. The expression*

$$U(x,y) = \frac{1}{2\pi r_3}\int_{C_2}\Phi ds - \frac{1}{\pi}\int_{C_3}\Phi\frac{\cos\phi}{d}ds \tag{13}$$

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* Goursat, *Cours d'Analyse Mathématique*, vol. 3, p. 222.

defines a function U which is harmonic at any point (x, y) exterior to C_3 and is such that as the point (x, y) approaches a point on C_3, $U(x, y)$ approaches the value of Φ at that point. In the above integrals d is the distance from (x, y) to a variable point M on C, and ϕ the angle betwen d and the radius of C_3 drawn to M.

By (7) the first of these integrals is zero, and hence

$$U(x,y) = -\frac{1}{\pi}\int_{C_3} \Phi \frac{\cos\phi}{d}\, ds = -\frac{1}{\pi}\int_0^{2\pi} \Psi \frac{\cos\phi}{d}\, d\theta.$$

Substituting the value of Ψ from (10) we obtain

$$(14) \qquad U(x,y) = -\frac{1}{\pi}\int_0^{2\pi} f_1 \frac{\cos\phi}{d}\, d\theta - \frac{1}{\pi}\int_0^{2\pi} f_2 \frac{\cos\phi}{d}\, d\theta.$$

But f_1 and f_2 are of constant sign and hence applying the first theorem of the mean for integrals to (14) we have

$$(15) \qquad U(x,y) = -\frac{1}{\pi}\left\{\left[m_1\frac{\cos\phi}{d}\right]\int_0^{2\pi} f_1\, d\theta + \left[m_2\frac{\cos\phi}{d}\right]\int_0^{2\pi} f_2\, d\theta\right\}$$

where $[m \cos(\phi/d)]$ indicates a mean of $\cos(\phi/d)$. But by (12), (15) becomes

$$U(x,y) = -\frac{1}{\pi}\left\{\left[m_1\frac{\cos\phi}{d}\right] - \left[m_2\frac{\cos\phi}{d}\right]\right\}\int_0^{2\pi} f_1\, d\theta.$$

Let $\overline{f_1}(r_3, \theta)$ indicate the maximum value of f_1 on C_3. Then

$$(16) \qquad |U(x,y)| \leqq 2\left|\left[m_1\frac{\cos\phi}{d}\right] - \left[m_2\frac{\cos\phi}{d}\right]\right| \overline{f_1}(r_3,\theta) \cdot$$

Suppose now that $\overline{f_1}$ approaches zero as r_3 approaches zero. Then, since for any point on C_1, $[m\cos(\phi/d)]$ is bounded, $|U(x, y)|$ can be made less than $\epsilon/4$ at all points on C_1 by taking r_3 sufficiently small, $r_3 = r_3'$, say. Now on C_3, Φ and U are equal, while on C_1, $\Phi = 0$. Hence in the region C_1C_3,

Φ will differ from U by less than $\epsilon/2$. Now consider any point (a, b) except P, interior to C_1. We can take r_3 so small that (a, b) lies in C_1C_3. Also by (16), r_3 can be taken so small, $r_3 = r_3''$ say, that $U(a, b) < \epsilon/2$. Hence by taking r_3 equal to the smaller of r_3' and r_3'' we have

$$|\Psi(a,b)| < \epsilon.$$

Since ϵ is any arbitrarily small number and (a, b) is any point in C_1 except P, we have the result that if $\overline{f_1}$ approaches zero as r_3 approaches zero, Ψ is identically zero in C_1.

Therefore, unless $\Phi \equiv 0$, there must be a mode of approach to P for which f_1 remains greater than some positive constant c_1, or in other words for which Φ remains greater than c_1/r. In the same manner we can prove that there exists a mode of approach to P for which Φ remains less than $-c_2/r$ where c_2 is a positive constant.

Suppose the constant c in (8) is positive. It is easy to prove that

$$\lim_{r\to 0}\left[c \log \frac{1}{r} - \frac{c_2}{r} \right] = -\infty$$

and hence for some mode of approach to P

$$\lim_{r\to 0}\left[c \log \frac{1}{r} + \Phi(x,y) \right] = -\infty .$$

In the same way, if c is negative, for some mode of approach to P

$$\lim_{r\to 0}\left[c \log \frac{1}{r} + \Phi(x,y) \right] = +\infty .$$

We can now formulate the following theorems.

THEOREM I. *In the neighborhood of an isolated singular point P, a harmonic function $f(x, y)$ is of the form*

$$f(x,y) = c \log \frac{1}{r} + \Phi(x,y) + V(x,y)$$

where c is a constant, r the distance from (x, y) to P, $\Phi(x, y)$, unless it is identically zero, harmonic in the neighorhood of P and such that there exist modes of approach to P for which the sum $c \log (1/r)+\Phi$ *tends towards plus infinity and also toward minus infinity; and V is harmonic everywhere in the neighborhood of P, including P.*

THEOREM II. *If a harmonic function $f(x, y)$ is bounded in the deleted neighborhood of an isolated singular point, it is in this neighborhood equal to a function which is harmonic in this neighborhood and also at P, and the singularity can be removed by defining $f(P)$ to be equal to* $\lim f(x, y)$ *as (x, y) approaches P.*

For the constant c and the function Φ must be identically equal to zero if f is bounded and we have $f = V$ in the neighborhood of P.

THEOREM III. *If a harmonic function tends toward plus infinity (minus infinity) in the neighborhood of an isolated singular point, it is of the form* $f = c \log (1/r) + V$ *where c is a positive (negative) constant and V is harmonic everywhere in this neighborhood, including P.*

For by Theorem I, Φ must be identically zero in this case.

THEOREM IV. *A harmonic function $f(x, y)$ cannot have an isolated singular point P which is such that for certain modes of approach to P, $f(x, y)$ tends toward plus infinity (minus infinity) and remains finite for all other modes of approach to P.*

For if $\Phi(x, y) \equiv 0$, we have the case described in Theorem III, while if $\Phi(x, y)$ is not identically zero, Theorem IV follows from Theorem I.

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