SOME RECENT WORK IN THE CALCULUS OF VARIATIONS*

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I. THE SIMPLEST PROBLEM.

1. *First Order Conditions.* The simplest problem of the calculus of variations is concerned with an integral of the form

$$(1) \qquad \int_{x_1}^{x_2} f(x, y, y')dx,$$

in which f is a function of class C'''† in a region R_1 of 3-space, determined by the conditions that (x, y) be in some region R of the xy-plane and y' be finite. The problem may then be formulated as follows.

To determine among all functions $y=y(x)$, $x_1 \leqq x \leqq x_2$, which (1) are of class C'; (2) satisfy the conditions $y(x_i)=y_i$, $i=1, 2$; and (3) are such that the points $(x, y(x))$ for $x_1 \leqq x \leqq x_2$ lie in the region R, a function $y_0(x)$ for which there exists a positive number d, such that all functions $Y(x)$, $x_1 \leqq x \leqq x_2$, which satisfy conditions (1), (2), and (3), and the further condition that $|Y(x)-y_0(x)| \leqq d$ for $x_1 \leqq x \leqq x_2$, give to the integral (1) a value not less (not greater) than the value which this integral has for $y_0(x)$. Here it is understood that y' denotes the derivative dy/dx. A function which satisfies the conditions (1), (2), and (3) is called an *admissible function.*

The classical procedure in solving this problem is as follows. We suppose that we have found a solution $y=y_0(x)$,

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† A function is said to be of class $C^{(n)}$ if it possesses continuous derivatives of orders $1, 2, \cdots, n$; a function is said to be of class $D^{(n)}$, if it is continuous, and consists of a finite number of parts of class $C^{(n)}$.

$x_1 \leqq x \leqq x_2$, of the problem. We then set up a *family of admissible functions*

$$y = y_0(x) + \epsilon\eta(x), \tag{2}$$

in which $\eta(x)$ and ϵ are so restricted as to make every function of the family (2) admissible (i. e., (i) η is of class C', (ii) $\eta(x_1) = \eta(x_2) = 0$, (iii) ϵ sufficiently small).* The integral determined for the functions (2) will then become a function of ϵ,

$$I(\epsilon) = \int_{x_1}^{x_2} f(x_1, y_0 + \epsilon\eta, y_0' + \epsilon\eta')dx, \tag{3}$$

which must possess a minimum for $\epsilon = 0$. Hence we obtain, as necessary conditions,

$$I'(0) = 0, \qquad I''(0) \geqq 0, \ (\leqq 0). \tag{4}$$

Conditions on the function $y_0(x)$ derived from (4_1) are called *first-order conditions*; those derived from (4_2) are called *second-order conditions*. It is readily found† that

$$I'(0) = \int_{x_1}^{x_2} (f_y\eta + f_{y'}\eta')dx, \tag{5}$$

in which $f_y = \partial f/\partial y$, $f_{y'} = \partial f/\partial y'$ and in which the arguments of f_y and of $f_{y'}$ are x, $y_0(x)$, $y_0'(x)$. The second term in (5) is now integrated by parts; thus we obtain by making use of condition (ii) on the admissible variation η, the condition that the function $y_0(x)$ must be such that

$$\int_{x_1}^{x_2} \eta\left(f_y - \frac{d}{dx} f_{y'}\right)dx = 0, \tag{6}$$

for every admissible variation η.

At this stage enters the *fundamental lemma of the calculus of variations*:

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* Functions $\eta(x)$ which satisfy conditions (i) and (ii) are called admissible variations.

† See, e. g., Bolza, *Lehrbuch der Variationsrechnung*, p. 21.

FUNDAMENTAL LEMMA. *If M is a continuous function of x, and if for every function η of class C' for which $\eta(x_1) = \eta(x_2) = 0$, we have $\int_{x_1}^{x_2} \eta M dx = 0$, then $M \equiv 0$ on (x_1, x_2).*

This lemma, which appears to have been proved for the first time by Stegemann in 1854, and the need for which does not seem to have been felt by earlier writers, is then applied to equation (6). Thus is obtained the *first necessary condition*, which a solution of the problem must satisfy, viz.,

$$f_y(x, y_0, y_0') - \frac{d}{dx} f_{y'}(x, y_0, y_0') = 0. \tag{I}$$

This equation is known as the *Euler equation*. A function which satisfies it is called an *extremal.*

2. *Second order conditions* Proceeding now to condition (4_2), we shall consider only the conditions for a *minimum.** It is then found that the condition

$$\int_{x_1}^{x_2} (f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2)dx \geqq 0 \tag{7}$$

must hold for every admissible variation, the arguments of f_{yy}, etc., being x, $y_0(x)$, and $y_0'(x)$. The classical methods for the further study of this condition are based upon a transformation of the integrand which shall make it possible to state conditions which will insure for it constancy of sign for all admissible variations η. We must refer the reader to the literature† for an account of the way in which (7) leads to the following two conditions:

$$f_{y'y'}(x, y_0(x), y_0'(x)) \geqq 0, \qquad x_1 \leqq x \leqq x_2, \tag{II}$$

(*Legendre's condition*);

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* Throughout this paper the minimum problems only will be considered; obvious modifications are necessary for the case of a maximum.

† See, e.g., Bolza, loc. cit., Chap. II; Kneser, *Lehrbuch der Variationsrechnung*, 2d ed., §21; Hadamard, *Leçons sur le Calcul des Variations*, p. 313–359.

(III) $$x_2 \leqq x_1',$$
(*Jacobi's condition*).

Here x_1' designates the first vanishing point after x_1 of a solution of the ordinary linear differential equation

(8) $$\left(f_{yy} - \frac{df_{yy'}}{dx}\right)u - \frac{d}{dx}(f_{y'y'}u') = 0,$$

which vanishes at x_1. Equation (8) in which f_{yy}, $f_{yy'}$ and $f_{y'y'}$ have the arguments x, $y_0(x)$, $y_0'(x)$ and are therefore functions of x, is known as *Jacobi's differential equation.* The condition (III) can be stated in different ways. Leaving aside the purely geometrical formulation, due to Kneser,* we shall state condition (III) in a form which introduces the important concept of a *field of extremals.* A *field* of extremals is a region S of the xy-plane such that through every one of its points (with the possible exception of a finite number of points) there passes a *unique* extremal. With this definition we can now replace (III) by the equivalent condition

(III'): $y=y_0(x)$, $x_1 \leqq x \leqq x_2$, must be such that there exists a field of extremals S which contains the curve determined by $y=y_0(x)$, $x_1 \leqq x \leqq x_2$, in its interior.

The slope of the unique extremal which passes through the point (x, y) of a field S is denoted by the symbol $p(x, y)$ called the *slope function* of the field. If a one-parameter family of extremals $y=\varphi(x, a)$, $a_1 \leqq a \leqq a_2$, furnishes a field S, it follows that the defining equation can be solved for a, whenever (x, y) is in S; the condition is usually imposed that this function $a(x, y)$ shall be of class C' in S. In this case it follows that the function $p(x, y)$ is given by the formula

$$p(x, y) = \varphi_x(x, a(x, y))$$

and that this function is then also of class C'.

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* See Bolza, loc. cit., § 13; Bliss, *Calculus of Variations*, p. 140 et seq.; Kneser, loc. cit., § 19.

3. *Weierstrass Condition.* Weierstrass showed that conditions (I), (II), (III) are not sufficient for a solution of the problem and obtained a further necessary condition. It will suit our purpose best to relate this condition to other results which were secured later.

Hilbert showed* that the integral

$$(9)\ I^{*} = \int_a^b \{f(x,y,p(x,y)) + (y' - p(x,y))f_{y'}(x,y,p(x,y))\}dx$$

taken along curves which lie entirely in a field S is *independent of the path.* It should be noticed that the integral (9) taken along an extremal is identical with the integral (1) along that extremal.

From this fact it now follows readily that if an extremal $C_0[y=y_0(x)]$ and an arbitrary admissible curve $C[y=y(x)]$ both lie in a field S, then

$$(10)\quad \Delta I = I_C - I_{C_0} = \int_{x_1}^{x_2} E(x,y(x),p(x,y(x)),y'(x))dx,$$

in which E is a function of four arguments introduced by Weierstrass and related to the function f of the problem by the formula

$$(11)\ E(x,y,u,v) \equiv f(x,y,v) - f(x,y,u) - (v - u)f_{y'}(x,y,u).$$

Formula (10), which expresses the Weierstrass Theorem, leads now to the *Weierstrass necessary condition.*†

(IV): *If* $y=y_0(x)$, $x_1 \leqq x \leqq x_2$ *is a solution of our problem, then we must have* $E(x, y_0, y_0', p) \geqq 0$, *for every finite value of* p.

It should be observed here that condition (IV) is not derived from (4_2); it is in fact entirely independent of the mode of reasoning upon which conditions (4) were based. Bolza‡ has shown that conditions (I), (II), (III), and (IV)

* See Bolza, loc. cit., p. 108.

† Bolza, loc. cit., §18.

‡ Loc. cit., pp. 116–119.

are not sufficient, and he derived a fifth necessary condition; this we shall not discuss here.

The theory proceeds at this point to set up sufficient conditions, chiefly by means of Weierstrass's theorem as expressed by (10). An important distinction should be introduced in the formulation of these conditions.

A solution of the problem stated in the beginning is said to furnish a *strong* extremum for the integral (1). If the admissible functions $Y(x)$ for which the value of the integral is compared with that along a solution $y_0(x)$ are restricted not merely by the condition

$$|Y(x) - y_0(x)| \leqq d, \tag{12}$$

but also by

$$|Y'(x) - y_0'(x)| \leqq d,$$

the solution is said to furnish a *weak* minimum. Sets of necessary and of sufficient conditions can now be formulated for a strong and for a weak minimum.*

4. *Various Extensions.* We consider now some modifications of the problem.

(a) The condition (2) on admissible functions may be replaced by the condition (2a), that the point $(x_1, y(x_1))$, or the point $(x_2, y(x_2))$, or both, lie on given curves. In this case we obtain as an additional first-order condition the requirement that the equation

$$f(x, y, y') + (s - y')f_{y'}(x, y, y') = 0$$

hold at the point in which the extremal meets the given curve, where s denotes the slope of the given curve at this point. This condition is known as the *transversality condition*; for the case that $f(x, y, y') \equiv g(x, y)\sqrt{1+y'^2}$, transversality reduces to orthogonality.

(b) If condition (1) on admissible functions be replaced by the condition (1a) that these functions be of class D', we obtain the first-order condition that at every point where a discontinuity in the derivative occurs (at such points

* See, e. g., Bolza, loc. cit., p. 127; Hadamard, loc. cit., p. 389, p. 397.

progressive and regressive derivatives y'^{+} and y'^{-} will exist), the following equations must hold:

$f_{y'}(x, y, y'^{+}) = f_{y'}(x, y, y'^{-})$, and $f - y'f_{y'}|^{+} = f - y'f_{y'}|^{-}$.

This condition, known as the corner-point condition, is due to Erdmann (1877).*

(c) Weierstrass considered the problem of determining a curve, $x = x(t)$, $y = y(t)$, $t_1 \leqq t \leqq t_2$, which will minimize the integral

$$\int_{t_1}^{t_2} F(x, y, x', y')dt, \tag{13}$$

where now the symbol $'$ denotes differentiation with respect to the parameter t. Leaving it to the reader to set up for the curve-problem an exact formulation analogous to the one given above for the function-problem, we may say that the Weierstrass form of the simplest problem of the calculus of variations (usually referred to as the parametric form) consists in the determination of a curve $x = x(t)$, $y = y(t)$, $t_1 \leqq t \leqq t_2$, which will furnish the extreme values of the integral (13). Conditions analogous to those mentioned for the function problem are arrived at in this case and we shall refer to these conditions freely, without explicitly stating them. It must be observed, however, that if this problem is to have any meaning at all, the integrand F must be such that the integral (13) shall depend on the *curve* only, and shall be independent of the particular parameter used for its representation by means of the function $x(t)$ and $y(t)$. This leads to the homogeneity condition,† according to which we must have

$$F(x, y, kx', ky') \equiv kF(x, y, x', y'), \tag{14}$$

whenever $k > 0$. From this condition on the function F follow important consequences, among which we mention that there must exist a function F_1 such that

$$F_{x'x'} = y'^2 F_1, \quad F_{x'y'} = -x'y'F_1, \quad F_{y'y'} = x'^2 F_1.$$

* See, e. g., Bolza, loc. cit., p. 367.

† See, Bolza, loc. cit., p. 193–195; Kneser, loc. cit., p. 11.

This function F_1 plays a role in the curve problem analogous to that played by $f_{y'y'}$ in the function problem.

5. *Recent Developments.* We are in a position now to mention some of the recent advances made in connection with the theory of the simplest problem.

(a) It was observed by Dubois-Reymond* in 1879 that the fundamental lemma of the calculus of variations (see §1, above) as applied to the integral (6), presupposes that $y''(x)$ exists and is continuous, which was not among the hypotheses laid down for the admissible curves. To meet the difficulty suggested by this remark, he proved the following generalization of the fundamental lemma. *If M is continuous and if for every function η of class C' for which $\eta(x_1)=\eta(x_2)=0$, we have $\int_{x_1}^{x_2}\eta' M\,dx=0$, then M is constant throughout* (x_1, x_2).

To apply this lemma, he integrated the first term in (5), so as to derive from (4_1) the condition that

$$\int_{x_1}^{x_2}\eta'\left(f_{y'}-\int_{x_1}^{x}f_y\,dx\right)dx=0, \tag{15}$$

for every admissible variation η; the Lemma of Dubois-Reymond then gives the condition

$$f_{y'}-\int_{x_1}^{x}f_y\,dx=\text{const.} \tag{16}$$

The conditions on the admissible functions and on the function f are sufficient to enable us to conclude from this that the derivative of $f_{y'}$ exists and is continuous on (x_1, x_2), and thus to obtain the Euler equation (I). It follows from this work that an admissible function which solves our problem must be of class C''.

Various further generalizations of the fundamental lemma in the direction suggested by Dubois-Reymond have since been made by others.† The most inclusive generalization

* See Bolza, loc. cit., p. 27.

† See Zermelo, MATHEMATISCHE ANNALEN, vol. 59 (1904), p. 558; Jacobstahl, ARCHIV DER MATHEMATIK UND PHYSIK, vol. 16 (1910), p. 82.

has been made recently by A. Haar,* who showed that if $v(x)$ is continuous, and such that for every function $u(x)$ of class $C^{(k)}$ for which $u^{(i)}(a)=u^{(i)}(b)=0$, $(i=0, \cdots, k-1)$, we have $\int_a^b vL(u)dx=0$, L being a linear homogeneous differential form of kth order whose coefficients are functions of x possessing derivatives of all orders and whose leading coefficient is unity, then $v(x)$ will be of class $C^{(k)}$ on (ab) and $\Lambda(v)=0$, where Λ is the adjoint of L. The proof is simple enough to be substituted for the usual proof of the fundamental lemma and of Dubois-Reymond's Lemma; these it contains as special cases for $L(u)=u$ and $L(u)=u'$ respectively.

Another way of dealing with the difficulty pointed out by Dubois-Reymond has been suggested by Razmadze,† who proved the following theorem. *If* $M(x)$ *and* $N(x)$ *are continuous on* (a, b), *and if for every function* $\eta(x)$ *of class* C' *such that* $\eta(a)=\eta(b)=0$, *we have* $\int_a^b(M\eta+N\eta')dx=0$, *then* N' *exists at every interior point of* (a, b), *and* $N'=M$. This theorem enables us to pass directly from equation (5) to the Euler equation, without integration by parts either of the second term or of the first term. This result of Razmadze has been further extended in an interesting paper by Kryloff‡ in such a way as to provide also for the case in which higher derivatives are involved. He proved that, if for every function η of class $C^{(n)}$ for which

$$\eta^{(i)}(a)=\eta^{(i)}(b)=0,\ i=1, \cdots, n-1,$$

we have

$$\int_a^b \sum_{j=0}^{n} M_j\eta^{(j)}dx = 0, \tag{17}$$

then we can conclude that

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* See ACTA LITTERARUM AC SCIENTIARUM R. U. H. FRANCIS-JOSEPHINAE, vol. 1 (1922), p. 33.

† See MATHEMATISCHE ANNALEN, vol. 84 (1921), p. 115.

‡ See BULLETIN DE L'ACADÉMIE DES SCIENCES DE L'OUARAÏNE (Classe des sciences physiques et mathématiques), vol. 1 (1923), p. 8.

$$\frac{d}{dx}\left\{\frac{d}{dx}\cdots\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{d}{dx}M_n - M_{n-1}\right) + M_{n-2}\right]\right.$$
$$\left. - M_{n-3} + \cdots + (-1)^{n-1}M_1\right\} + (-1)^n M_0 = 0,$$

provided the functions M_j are such that the successive derivatives which appear in this formula exist. If we suppose that the functions M_j are of class $C^{(j)}$, then the last conclusion may be written in the form

$$\sum_{j=0}^{n}(-1)^j M_j^{(j)} = 0.$$

It is this result which I have had in my possession for a number of years, with the intention of using it in the study of a generalization of the simplest problem to which reference will be made in the sequel. In this result, the equation (17) is supposed to hold for every function η of class $C^{(n)}$ which satisfies two general linear homogeneous boundary conditions of the form

$$\sum_{j=0}^{n}\alpha_{ij}\eta^{(j)}(a) + \beta_{ij}\eta^{(j)}(b) = 0, \qquad (i = 1, \cdots, 2n),$$

α_{ij} and β_{ij} being constants.

(b) We have seen that conditions (II) and (III) are obtained in the classical theory of the simplest problem from the condition (4_2). It has also been mentioned that Kneser has developed a geometrical formulation of (III). In the study of more complicated problems, and even in connection with the simplest problem in parametric form, it was found that the transformation of the second variation, analogous to the one referred to under §2, became very complicated. The tendency has been therefore to follow Kneser's geometric method, which indeed is capable of extension to such problems in a very elegant manner. This, however, has the drawback of leaving unsettled exceptional cases, arising out of possible singularities of the curves involved. To meet the difficulty which arises in this way, Bliss has developed within recent years a new method of obtaining the second-order conditions from the second variation. This method has been applied by him and by several of his pupils to a number of problems.* It is easy to explain its essential features in connection with the problem with which we have thus far been concerned. The condition (4_2) has been said to lead to condition (7). If we denote the integrand in (7) by $2\Omega(\eta, \eta')$, i.e.,

$$(18) \qquad 2\Omega(\eta, \eta') \equiv f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2,$$

condition (7) may be interpreted as requiring that 0 shall be the minimum value of the integral $\int_{x_1}^{x_2} \Omega(\eta, \eta')dx$. Hence any admissible variation $\eta(x)$ which gives this latter integral the value 0 must satisfy the conditions necessary for an extremal of this integral, looked upon as an integral in the $x\eta$-space. By applying to the integral of (18) the necessary conditions for a minimum which do not depend on (4_2), the second-order conditions may be obtained. It is seen that the Euler equation obtained from the function (18) is identical with the Jacobi equation (8). By applying the Weierstrass condition to the function $\eta \equiv 0$, which gives the integral of (18) its minimum value and hence must satisfy the necessary conditions, one obtains the Legendre condition for the original integral. Bliss shows furthermore that if the Jacobi equation (8) possesses an integral which vanishes at x_1 and again at a point $x_1' < x_2$, then it is possible to construct a function of class D' which gives the integral $\int \Omega dx$ its minimum value 0, but which fails to satisfy the corner-point condition; it would therefore be possible to render this integral negative. Thus the necessity of condition (III) is established.

(c) We turn next to another extension of the simplest problem. In a recent paper, Razmadze† has investigated

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* See Bliss, this BULLETIN, vol. 26 (1920), p. 343, and TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 195. Also D. M. Smith, TRANSACTIONS, vol. 17 (1916), p. 459; M. B. White, TRANSACTIONS, vol. 13 (1912), p. 175; G. A. Larew, TRANSACTIONS, vol. 20 (1919), p. 1; F. LeStourgeon, TRANSACTIONS, vol. 21 (1920), p. 357.

† See MATHEMATISCHE ANNALEN, vol. 94 (1925), p. 1.

the conditions which must be satisfied by a *discontinuous curve* which furnishes a minimum for the integral (1). He supposes that a solution for this problem is given by a curve which is of class C' throughout the interval (x_1, x_2) except at one point x_0, where it has a finite discontinuity.

The importance of this question becomes clear when one considers the problem of Weierstrass, in which it is required to minimize the integral $\int_{-1}^{+1} x^2y'^2dx$ by a function $y(x)$, for which $y(-1)=a$, $y(1)=b$, $a\neq b$.* The Euler equation for this problem becomes $x^2y'=\text{const.}$ which has, besides the singular integral $x=0$, the general integral $y=c_1/x+c_2$. Hence it is clear that there exists no continuous extremal joining the given points. On the other hand, it is readily seen that the greatest lower bound of the values of the integral for the continuous functions joining the given points is 0, when one considers the family of functions C_e given by the equation

$$y = \frac{a+b}{2} + \frac{(b-a)}{2}\,\frac{\text{arc tan }(x/e)}{\text{arc tan }(1/e)},$$

for which the value of the integral becomes

$$J(e) < \frac{e(b-a)^2}{2\text{ arc tan }(1/e)},$$

so that $J(e)\to 0$, when $e\to 0$. When $e\to 0$, the function C_e tends toward the discontinuous function given by the equations

$$\begin{aligned} y &= a \quad &&\text{for} \quad -1 \leqq x < 0,\\ y &= (a+b)/2 \quad &&\text{for} \quad x = 0,\\ y &= b \quad &&\text{for} \quad 0 < x \leqq 1; \end{aligned}$$

for this the integral does indeed take its minimum value 0.

What are the conditions which a curve with a finite discontinuity must satisfy if it is to minimize the integral (1)? The first-order conditions are found to be, besides the

* See Bolza, loc. cit., p. 420.

requirement that the continuous parts of the curve should be extremals, the equations

$$f(x_0,y_0,y_0')=f(x_0,\bar{y}_0,\bar{y}_0'),\quad f_{y'}(x_0,y_0,y_0')=f_{y'}(x_0,\bar{y}_0,\bar{y}_0')=0,$$

where (x_0, y_0) and $(x_0, \bar{y}_0)$ are the two points between which the finite discontinuity takes place, and y_0' and $\bar{y}_0'$ are the slopes of the parts of the curve at these points. These conditions are necessary in the general case, in which discontinuous comparison curves can be admitted whose break *need not* occur for the same value of x as gives the cut-point for the solution. In the exceptional case in which the cut-points of all admissible discontinuous comparison curves have the same abscissa, the first of the above conditions is replaced by a condition which determines the abscissa of the cut-point. A further necessary condition is that if y_k lies between y_0 and $\bar{y}_0$, and y_k' is arbitrary, we must have $f(x_0, y_k, y_k')\geqq f(x_0, y_0, y_0')$ and $f(x_0, y_k, y_k')\geqq f(x_0, \bar{y}_0, \bar{y}_0')$.

Assuming further that the two continuous parts of the curve satisfy the conditions of Legendre, Jacobi, and Weierstrass for continuous solutions of the problem, Razmadze obtains a theory of conjugate points, further necessary conditions, and also sufficient conditions. For the detailed results the reader is referred to Razmadze's paper.

(d) In 1907, Bliss introduced a new form for the simplest problem of the calculus of variations, by asking for a curve such that

$$\int_{t_1}^{t_2} F(x,y,\theta)dt$$

becomes a minimum, in which θ is defined by the equations*

$$\cos\theta = \frac{x'}{\sqrt{x'^2+y'^2}},\quad \text{and}\quad \sin\theta = \frac{y'}{\sqrt{x'^2+y'^2}}\cdot$$

The theory is in some respects simpler than in the ordinary Weierstrass formulation, inasmuch as the quantities x, y, θ are independent of the choice of parameter.

* See TRANSACTIONS OF THIS SOCIETY, vol. 8 (1907), p. 405.

This method of treatment has since been extended to spaces of three and more dimensions by Rider* and Sakellariou,† who have also developed the theory of the Hilbert independent integral and the Weierstrass E-function for integrals of such form.

(e) For some years I have had under consideration a modification of the simplest problem of the calculus of variations, obtained by the consideration of more general boundary conditions. The problem may roughly be formulated as follows:

To minimize the integral (1) by a function which satisfies two linear boundary conditions of the form

$$\alpha_{i1}y(a) + \alpha_{i2}y(b) + \beta_{i1}y'(a) + \beta_{i2}y'(b) = C_i, \qquad i = 1,2.$$

It is for the purpose of this problem that the extension of the fundamental lemma mentioned in §5(a) was developed.

II. The General Lagrange Problem and the Mayer Problem

1. *Introduction.* A variety of problems of the calculus of variations is included in the *Lagrange problem*, which is concerned with an integral of the form

$$\text{(19)} \qquad \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx,$$

where f is a function of class C''' with respect to its $2n+1$ arguments in a region R_1 of $(2n+1)$-space, determined by the conditions that $(x, y_1, \cdots, y_n)$ be in a certain region R of $(n+1)$-space and that‡ y_i', $i=1, \cdots, n$ be finite. The problem consists in determining among all sets of n functions

* See Tôhoku Mathematical Journal, vol. 13 (1918), p. 165; Washington University Studies, vol. 5 (1918), p. 97; American Journal of Mathematics, vol. 39 (1917), p. 241.

† See Tôhoku Mathematical Journal, vol. 13 (1918), p. 15; Annali di Matematica, vol. 28 (1919), p. 169; Palermo Rendiconti, vol. 44 (1920), p. 53.

‡ The indices i, j, and k will be understood to run over the range $1, \cdots, n$ throughout the sequel, unless differently indicated.

$y_i(x)$, $x_1 \leqq x \leqq x_2$, which (1) consist of functions of class C', (2) satisfy the conditions $y_i(x_1) = y_{i1}$, $y_i(x_2) = y_{i2}$, (3) are such that the points $(x, y_1, \cdots, y_n)$ for $x_1 \leqq x \leqq x_2$ lie in the region R, (4) satisfy the differential equations.

$$(20) \qquad \varphi_\mu(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx = 0,$$

for $\mu = 1, \cdots, m < n$, a set of functions $y_{i0}(x)$ for which there exists a positive number d, such that for all sets of functions $Y_i(x)$, $x_1 \leqq x \leqq x_2$, which satisfy conditions (1) to (4), and the further conditions that $|Y_i(x) - y_{i0}(x)| < d$ for $x_1 \leqq x \leqq x_2$, the integral (19) has a value not less than (not greater than) the value which it has for $y_{i0}(x)$. A set of functions which fulfils conditions (1) to (4) is said to determine an *admissible curve in* $(n+1)$-space for this problem.

A second general formulation which includes a large number of problems of the calculus of variations is given by the *problem of Mayer*, which may be stated as follows. To determine among all sets of functions $y_i(x)$, $x_1 \leqq x \leqq x_2$, which (1) consist of functions of class C'; (2) satisfy the conditions $y_i(x_1) = y_{i1}$, $y_\nu(x_2) = y_{\nu 2}$, $\nu = 2, \cdots, n$; (3) are such that the points $(x, y_1, \cdots, y_n)$ for $x_1 \leqq x \leqq x_2$ lie in a given region R of the xy-plane; (4) satisfy the differential equations $\varphi_\mu(x, y_1, \cdots, y_n; y_1', \cdots, y_n') = 0$, $\mu = 1, \cdots, m < n$, a set of n functions $y_{i0}(x)$ for which there exists a positive number d, such that, if $Y_i(x)$ be any other set of functions, which satisfies conditions (1) to (4), and the further condition that $|Y_i(x) - y_{i0}(x)| < d$ for $x_1 \leqq x \leqq x_2$, then $Y_i(x_2)$ shall be not less than (not greater than) $y_{i0}(x_2)$. The theory of these problems, their interrelations and their importance in the study of general problems of the calculus of variations have formed the subject of many studies.* The first step in those studies consists in establishing a problem in which the admissible curves are restricted by conditions (1), (2), and (3) only, and which is, at least as far as first-

*See, e.g., Bolza, loc. cit., p. 543 and p. 573; Hadamard, loc. cit., p. 217–224, p. 176; Kneser, loc. cit., Chap. 6.

order conditions are concerned, equivalent to the given problem. This is accomplished by means of the *Euler-Lagrange multiplier rule,** which for the Lagrange problem may be stated as follows. If $y_i = y_{i0}(x)$ furnish a solution for the general Lagrange problem, then there must exist m functions $\lambda_\mu(x)$ such that y_{i0} and λ_μ satisfy the first order conditions for a minimum of the integral

$$\int_{x_1}^{x_2} F(x, y_i, y_i', \lambda_\mu) dx, \tag{21}$$

the class of admissible functions being determined by conditions (1), (2), (3), and where

$$F = f + \sum \lambda_\mu \varphi_\mu. \tag{22}$$

For the Mayer problem a similar theorem has been proved. This multiplier-rule has been established also for problems in which to the condition (4) is adjoined a condition requiring that the unknown functions $y_i(x)$ also satisfy a number of finite equations. Difficulties of a particular kind, clearly recognized by Hahn,† had to be overcome in such cases. Other difficulties arise when the conditions (2) are replaced by more general conditions as to the end-points. These circumstances have given rise to a thorough working over of the subject, in the course of which further extensions and other valuable results have been secured. Mention must be made of a very elegant and complete treatment of this subject given by Bliss in a course of lectures at the University of Chicago, during the summer of 1925.‡

2. *Extensions and Generalizations.* We proceed to report briefly on some of the recent treatments of the Euler-Lagrange rule.

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* See Bolza, loc. cit., § 66–72.

† Mathematische Annalen, vol. 58 (1903), p. 152; see also Bolza, loc. cit., p. 564, where further references are given.

‡ I acknowledge with thanks the opportunity which Professor Bliss has given me to see a mimeographed copy of these lectures, prepared by Mr. O. E. Brown, Northwestern University.

(a) In the second of two papers, devoted to this subject, Bolza* has considered the following parametric problem.† To minimize the expression

$$U = \int_{t_0}^{t_1} f(y_1, \cdots, y_n, y_1', \cdots, y_n')dt + G(y_{10}, \cdots, y_{n0}, y_{11}, \cdots, y_{n1}),$$

when the admissible curves are determined by functions

$$y_i = y_i(t), \qquad t_0 \leqq t \leqq t_1,$$

which satisfy the differential equations

$$\varphi_\alpha(y_i, y_i') = 0, \qquad \alpha = 1, \cdots, p,$$

and the finite equations

$$\psi_\beta(y_i) = 0, \qquad \beta = 1, \cdots, q,$$

while the end-points y_{i0}, y_{i1} satisfy the equations

$$\chi_\gamma(y_{i0}, y_{i1}) = 0, \qquad \gamma = 1, \cdots, r;$$

it is moreover stipulated that along the minimizing curve $y_i = y_{i0}(x)$, the determinant

$$(23) \qquad \begin{vmatrix} \dfrac{\partial\varphi_\alpha}{\partial y_1'}, & \cdots, & \dfrac{\partial\varphi_\alpha}{\partial y_m'} \\ \dfrac{\partial\psi_\beta}{\partial y_1}, & \cdots, & \dfrac{\partial\psi_\beta}{d y_m} \end{vmatrix} \neq 0, \qquad m = p + q < n,$$

and also that the matrix

$$(24) \qquad \left\| \begin{matrix} \dfrac{\partial\psi_\beta}{\partial y_1}\Big|_{y_{i0}}, & \cdots, & \dfrac{\partial\psi_\beta}{\partial y_n}\Big|_{y_{i0}}, & 0, & \cdots, & 0 \\ 0, & \cdots, & 0, & \dfrac{\partial\psi_\beta}{\partial y_1}\Big|_{y_{i1}}, & \cdots, & \dfrac{\partial\psi_\beta}{\partial y_n}\Big|_{y_{i1}} \\ \dfrac{\partial\chi_\gamma}{\partial y_{10}}, & \cdots, & \dfrac{\partial\chi_\gamma}{\partial y_{n0}}, & \dfrac{\partial\chi_\gamma}{\partial y_{11}}, & \cdots, & \dfrac{\partial\chi_\gamma}{\partial y_{n1}} \end{matrix} \right\|,$$

* See MATHEMATISCHE ANNALEN, vol. 64 (1907), p. 370; ibid., vol. 74 (1913), p. 430; see also Hilbert, MATHEMATISCHE ANNALEN, vol. 62 (1906), p. 351.

† To save space, I shall only give a rough statement of the different problems that are discussed; by comparing with the more complete statements given on pages 475 and 484, the reader will have little difficulty in supplying exact formulations.

of $2q+r$ rows and $2n$ columns shall be of rank $2q+r$ for the arguments furnished by the end-points of the minimizing curve.

It is clear that this problem reduces to the most general Lagrange problem with general boundary conditions, if $G\equiv 0$, and to the Mayer problem with general boundary conditions if $f\equiv\psi_\beta\equiv 0$, and $G\equiv y_{11}$.

Bolza proved that if $y_i=Y_i(t)$, $i=1, \cdots, n$, furnishes a solution for this problem, then there exist $p+q$ functions $\lambda_\alpha(t)$ and $\mu_\beta(t)$, and also $2q+r+1$ constants l_0, l_β^0, l_β^1, $l_{q+\gamma}$ such that the functions Y_i, λ_α and μ_β satisfy the system of n differential equations

$$\frac{\partial\Omega}{\partial y_i}-\frac{d}{dt}\frac{\partial\Omega}{\partial y_i'}=0,$$

and the $2n$ boundary conditions

$$H_i^0=0,\quad H_i^1=0.$$

Here

$$\Omega=l_0f+\sum\lambda_\alpha\varphi_\alpha+\sum\mu_\beta\psi_\beta,$$

and

$$H_i^\epsilon=(-1)^{\epsilon+1}\left.\frac{\partial\Omega}{\partial y_i'}\right|_{t_\epsilon}+l_0\frac{\partial G}{\partial y_{i\epsilon}}+\sum l_\beta^\epsilon\left.\frac{\partial\psi_\beta}{\partial y_i}\right|_{t_\epsilon}$$
$$+\sum l_{q+\gamma}\frac{\partial\chi_\gamma}{\partial y_{i\epsilon}},\qquad \epsilon=0,1,$$

and the constants l_0, $l_{q+\gamma}$ and $l_\beta=l_\beta^0+l_\beta^1+\int_{t_0}^{t_1}\mu_\beta\,dt$ do not all vanish.

This result holds both for the *normal* and for the *abnormal* cases; these are obtained for $l_0\neq 0$ and $l_0=0$, respectively. Furthermore, the usual theorems as to uniqueness of the constant and of the functional multipliers are established for the normal case; and the abnormal case receives a simple characterization. The difficulties hinted at as arising from the finite conditions $\psi_\beta(y_i)=0$ are obviated by means of the condition on the matrix (24), which insures against redundancy in the conditions arising from the finite equations and from the end-point equations.

(b) Another general form of the Euler-Lagrange rule has been given by Bliss,* who treated the following problem:

To determine a set of functions of class C'', $y_i = y_i(x)$, $x_1 \leqq x \leqq x_2$, which satisfy the differential equations (20), and the boundary conditions

$$f_{\gamma'}(x_1, y_{i1}, x_2, y_{i2}) = 0, \qquad \gamma' = 2, \cdots, r,$$

while the function $f_1(x_1, y_{i1}, x_2, y_{i2})$ is minimized.

It is an important feature of this formulation that the condition (23) is replaced by the less restrictive one that the matrix $||\partial\varphi_\mu/\partial y_i'||$ shall everywhere be of rank m. By proving that a matrix of mn continuous functions and of rank m at every point of (x_1, x_2), can always be extended into a determinant of order n, which vanishes nowhere on (x_1, x_2), Bliss succeeds in accomplishing with this more general condition the same purpose for which a condition like (23) had always been introduced, viz., that of setting up a family of admissible comparison curves. The further requirement is made that the matrix

$$\left\| \frac{df_\gamma}{\partial x_1} \quad \frac{df_\gamma}{dy_{i1}} \quad \frac{df_\gamma}{dx_2} \quad \frac{df_\gamma}{dy_{i2}} \right\|, \qquad \gamma = 1, \cdots, r,$$

of r rows and $2n+2$ columns shall be of rank r at the end-points of the minimizing curve; this is seen to be analogous to the earlier condition on the matrix (24). This general problem, like the one mentioned under (a), includes most of the problems of the calculus of variations, which are concerned with simple integrals, and the interrelations are pointed out, although not completely. For the present problem, the following result is obtained:

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* See TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 305.

If the functions $Y_i(x)$ furnish a solution, then there exist m functions λ_μ of class C', not all identically 0 on (x_1, x_2), such that the functions Y_i and λ_μ satisfy the differential equations

$$\frac{\partial \Omega}{\partial y_i} - \frac{d}{dx}\frac{\partial \Omega}{\partial y_i'} = 0,$$

and the boundary conditions, which are expressed by the requirement that the rank of the matrix

$$\left\| \begin{array}{cccc} \dfrac{\partial f_\gamma}{\partial x_1}, & \dfrac{\partial f_\gamma}{\partial y_{i1}}, & \dfrac{\partial f_\gamma}{\partial x_2}, & \dfrac{\partial f_\gamma}{\partial y_{i2}} \\ \Omega(x_1) - \Upsilon_1, & \left.\dfrac{\partial \Omega}{\partial y_i'}\right|_{x_1}, & -\Omega(x_2) + \Upsilon_2, & -\left.\dfrac{\partial \Omega}{\partial y_i'}\right|_{x_2} \end{array} \right\|$$

of $r+1$ rows and $2n+2$ columns shall be less than $r+1$; here $\Omega = \Sigma\lambda_\mu\varphi_\mu$, and

$$\Upsilon_1 = \sum y_i' \left.\frac{\partial \Omega}{\partial y_i'}\right|_{x_1}, \qquad \Upsilon_2 = \sum y_i' \left.\frac{\partial \Omega}{\partial y_i}\right|_{x_2}.$$

(c) An apparently very wide generalization of the Euler-Lagrange rule has recently been given by Hahn,* who considers the following problem. Suppose given r functional operators

$$W_\gamma(y_i, a, b), \qquad \gamma = 1, \cdots, r,$$

each of which associates a real number with every set of n functions $y_i(x)$ of class C', $a \leqq x \leqq b$, which satisfies the conditions

$$(25) \qquad | y_i(x) - y_{i0}(x) | < h, \ | y_i'(x) - y_{i0}'(x) | < h,$$

$$| a - a_0 | < h, \ | b - b_0 | < h,$$

where $y = y_{i0}(x)$, $a_0 \leqq x \leqq b_0$, is some particular set of functions of class C', and where h is a positive constant. Furthermore, let

$$\Phi_\mu(y_i, a, b, t), \qquad \mu = 1, \cdots, m,$$

* See WIENER BERICHTE, vol. 131 (1922), p. 531.

define a similar operator for every t on (t_1, t_2). It is supposed that

(1) If

$$y_i(x) = y_{i0}(x) + \sum_{\sigma=1}^{s} \epsilon_\sigma \eta_{i\sigma}(x) + (\epsilon), \tag{26}$$

$$y_i'(x) = y_{i0}'(x) + \sum_{\sigma=1}^{s} \epsilon_\sigma \eta_{i\sigma}'(x) + (\epsilon),$$

$$a = a_0 + \sum_{\sigma=1}^{s} \epsilon_\sigma \alpha_\sigma + (\epsilon),$$

$$b = b_0 + \sum_{\sigma=1}^{s} \epsilon_\sigma \beta_\sigma + (\epsilon),$$

where (ϵ) represents a function of the parameters $\epsilon_1, \cdots, \epsilon_s$ which tends to zero with these parameters, and where $\eta_{i\sigma}$, α_σ, and β_σ are so chosen that the conditions (25) are satisfied, then

$$W_\gamma(y_i, a, b) = W_\gamma(y_{i0}, a_0, b_0) + \sum_\sigma \epsilon_\sigma V_\gamma(\eta_{i\sigma}, \alpha_\sigma, \beta_\sigma) + (\epsilon)\ ,$$

and

$$\Phi_\mu(y_i, a, b, t) = \Phi_\mu(y_{i0}, a_0, b_0, t) + \sum_\sigma \epsilon_\sigma \Psi_\mu(\eta_{i\sigma}, \alpha_\sigma, \beta_\sigma, t) + (\epsilon),$$

in which $V_\gamma(\eta_i, \alpha, \beta)$ and $\Psi_\mu(\eta_i, \alpha, \beta, t)$ are linear continuous functional operators* with respect to η_i and η_i', and linear continuous functions of α, β and of α, β, t, respectively.

(2) If s sets of functions $\eta_{i\sigma}(x)$ of class C'' on (a_0, b_0) and s sets of constants α_σ, β_σ are given which satisfy the linear functional equations $\Psi_\mu(\eta_i, \alpha, \beta, t) = 0$, then there exists for every set of values of the parameters $\epsilon_1, \cdots, \epsilon_s$, sufficiently small in absolute value, a solution y_i, a, b, of the functional equations $\Phi_\mu(y_i, a, b, t) = \Phi_\mu(y_{i0}, a_0, b_0, t)$, of the form (26).

(3) For every $e > 0$, there exists a d, such that if $\psi_\mu(t)$ are continuous functions of t on (t_1, t_2), for which $|\psi_\mu(t)| < d$,

* For the definitions of these terms, see, e.g., Lévy, *Leçons d'Analyse Fonctionnelle*, pp. 50, 52.

then there exists at least one set of functions $\eta_i(x)$ of class C' on (a_0, b_0) and a pair of constants α, β, such that

$$|\eta_i| < e, \qquad |\eta_i'| < e, \qquad |\alpha| < e, \qquad |\beta| < e,$$

and such that

$$\Psi_\mu(\eta, \alpha, \beta, t) \equiv \psi_\mu(t).$$

The following problem is now considered. Among all sets of n functions $y_i(x)$ of class C' which satisfy the conditions (25) and for which

$$W_{\gamma'}(y_i, a, b) = 0, \qquad \gamma' = 2, \cdots, r$$

and

$$\Phi_\mu(y_i, a, b, t) = 0, \qquad \mu = 1, \cdots, m; \qquad t_1 \leqq t \leqq t_2,$$

to determine one which will minimize the operator $W_1(y_i, a, b)$.

It is shown in the first place that there must exist r numbers $l_1, \cdots, l_r$ not all equal to 0, such that

$$V(\eta_i, \alpha, \beta) \equiv \sum l_\gamma V_\gamma(\eta_i, \alpha, \beta) = 0$$

for every solution η_i, α, β of the linear functional equations

$$\Psi_\mu(\eta, \alpha, \beta, t) = 0.$$

Next it is shown that if

$$\Psi_\mu(\eta_i, \alpha, \beta, t) = \Psi_\mu(\zeta_i, \gamma, \delta, t),$$

then we will also have

$$V(\eta_i, \alpha, \beta) = V(\zeta_i, \gamma, \delta),$$

so that the value of $V(\eta_i, \alpha, \beta)$ depends only upon $\Psi_\mu(\eta, \alpha, \beta, t)$ which are ordinary functions ψ_μ of t. We may then write

$$V(\eta_i, \alpha, \beta) = U(\psi_1, \cdots, \psi_m),$$

where U is a functional operator. This operator is then proved to be linear and continuous. Hence, by the use of a theorem of Riesz,* of which Hahn gives a very elegant

* See, e.g., Lévy, loc. cit., p. 55; also footnote (2) on first page of Hahn's paper mentioned above, in which full references are given for this theorem.

proof in this paper, we conclude that U can be represented as a Stieltjes integral, i.e., that there exist functions of limited variation $v_\mu(t)$, such that

$$U = \sum_{\mu=1}^{m} \int_{t_1}^{t_2} \psi_\mu dv_\mu(t).$$

This result combined with the preceding results, leads to the following theorem.

If the set of functions y_i furnishes a solution of the problem, then there exist r constants l_γ not all 0 and m functions of limited variation λ_μ such that

$$\sum_\gamma l_\gamma V_\gamma(\eta_i,\alpha,\beta) + \int_{t_1}^{t_2} \sum_{\mu=1}^{m} \Psi_\mu(\eta_i,\alpha,\beta,t)d\lambda_\mu(t) = 0$$

for every set of functions η_i, of class C' in (a_0, b_0), and for all numbers α, β.

This is the Euler-Lagrange rule for this general minimum problem, which includes as special cases the problems mentioned under (a) and (b).

3. *Second Order Conditions.* Other advances made in the theory of the Lagrange problem and of the Mayer problem have to do with second order conditions and with extensions to these problems of the Hilbert independent integral and of the Weierstrass theorem, which were mentioned in connection with the simplest problem of the calculus of variations in I, §3. The first point of importance is that whereas the Hilbert integral, for the simplest problem, is independent of the path of integration in *any* field of extremals, the independence of the analogous integral for these more general problems takes place only in the "Mayer fields," characterized by special conditions. Returning to the Lagrange problem, as formulated in §1, we suppose we have an n-parameter family of extremals which furnishes a field, and we define, as in the case of the simplest problem, slope functions of the field $p_i(x,y_1, \cdots, y_n)$, and also functions $q_\mu(x,$

$y_1, \cdots, y_n$) which are obtained as the Lagrange multipliers for the unique extremal which passes through the point. Then it is shown that the integral

$$I^* = \int_{x_1}^{x_2} \left\{ f(x, y_i, p_i) - \sum_i (p_i - y_i') F_{y_i'}(x, y_i, p_i, q_\mu) \right\} dx,$$

in which F is the function defined by (22), will be independent of the path provided the following $n(n-1)/2$ relations are satisfied:

$$\frac{\partial F_{y_k'}(x, y_i, p_i, q_\mu)}{\partial y_h} = \frac{\partial F_{y_h'}(x, y_i, p_i, q_\mu)}{\partial y_k}, \qquad h, k = 1, \cdots, m.$$

These conditions determine the special fields for which the Hilbert independence property holds and which are known as *Mayer fields*. The construction of Mayer fields, the Weierstrass theorem for Mayer fields, the connection with the transformation of the second variation and with second-order conditions have formed the subject of a number of papers.

(a) The conditions for the independence of the Hilbert integral, both for the case when all curves in a field are considered (absolute invariance) and for the case, suggested by Radon,* in which only curves which satisfy the equations (20) are taken into account (relative invariance), were studied by Bolza,† who also gave a simple geometric interpretation for these conditions, upon which the construction of Mayer fields can be based. He showed that, at least for the special Lagrange problem, to which the problem of minimizing the integral $\int f(x, y, y', \cdots, y^{(n)})$ can be reduced,‡ the relative invariance of the Hilbert integral carries the absolute invariance as a consequence.

The classical treatment of the second variation by Clebsch, Mayer, and von Escherich was based on the

* See Wiener Berichte, vol. 119 (1910), p. 1257.

† See Palermo Rendiconti, vol. 31 (1911), p. 257, and vol. 32 (1911), p. 111.

‡ See Bolza, *Vorlesungen*, p. 543.

transformation of the second variation to which we have had occasion to refer earlier. The complicated character of these transformations constitutes a serious drawback to this treatment of the second variation. A valuable contribution to the problem of relating these transformations to other parts of the theory was made by Hahn.* A considerable advance was made through the application to the second variation for the integral (19) of the method of Bliss, the essential features of which were discussed in I, §5(b).

(b) This was done by D. M. Smith† who obtained the analogues of the Legendre and Jacobi conditions, by applying the Weierstrass condition and the corner-point conditions to the secondary minimum problem. In a paper presented to the Toronto Congress in 1924, Bliss carried the treatment of the second variation still further by showing how Mayer fields may be constructed for the secondary problem, then using the Hilbert independent integral, and finally deriving the Weierstrass formula for the total variation of the secondary problem in terms of the E function. By taking an arbitrary admissible variation η_i for one curve and the curve $\eta_i \equiv 0$ for the other, an expression for the second variation is then obtained, in a very simple manner, which turns out to be exactly the one sought for by the complicated transformations of the earlier theories.

The same form of treatment was carried through for the Mayer problem with fixed end-points by Gillie A. Larew‡ again leading to the analogs of the Legendre and Jacobi conditions. In a later paper§ the same author set up a definition of an invariant integral and a definition of Mayer fields for the Mayer problem. This made it possible to prove a theorem analogous to the Weierstrass theorem on

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* See PALERMO RENDICONTI, vol. 29 (1910), p. 49.

† See TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 459.

‡ See TRANSACTIONS OF THIS SOCIETY, vol. 20 (1918), p. 1.

§ Ibid., vol. 26 (1924), p. 61.

the total variation. Space is lacking for a discussion of the details of these papers.

III. Multiple Integrals

1. *Parametric Form.* The study of the minima of multiple integrals has not reached the advanced stage which has been attained in the theory of the simple integral. The fundamental problem is concerned with an integral of the form

$$(27) \qquad \int_{(n)} f(x_1, \cdots, x_n, z, z_1, \cdots, z_n) dx_1 \cdots dx_n,$$

in which $z_i = \partial z / \partial x_i$, and in which the integral is to be extended over a closed manifold M in the n-space determined by the coordinates $x_1, \cdots, x_n$. The function f is of class C''' in a domain R_1 of $(2n+1)$-space determined by the conditions that $(x_1, \cdots, x_n, z)$ be in a domain R of $(n+1)$-space and that z_i be finite. The question is then to determine among all functions z of $x_1, \cdots, x_n$ which (1) are of class C', (2) assume on the boundary of M preassigned values, (3) yield points $(x_1, \cdots, x_n, z(x_1, \cdots, x_n))$ which lie in R, when $(x_1, \cdots, x_n)$ is in M, a function Z for which there exists a positive number d, such that for any function z, which satisfies conditions (1), (2), (3), and also the condition that $|Z - z| < d$ for $(x_1, \cdots, x_n)$ in M, the integral (27) has a value not less (greater) than the value which it takes for Z.

By the same procedure which led to the Euler equation in the simple problem, one reaches the conclusion that the function z must satisfy the partial differential equation

$$f_z - \sum \frac{\partial}{\partial x_i} f_{z_i} = 0.$$

A more symmetric form for this problem, and one which connects more readily with the theory of surfaces, is obtained when parametric representation is introduced. We have then to consider the minimum of an integral

$$\int_{(n-1)} F(x_i, x_{i\mu})du_1 \cdots du_{n-1}, \tag{28}$$

for $i=1, \cdots, n$, and $\mu=1, \cdots, n-1$, in which $x_{i\mu}=\partial x_i/\partial u_\mu$ and in which the integral is to be extended over a closed manifold in the $(n-1)$-space of the parameters $u_1, \cdots, u_{n-1}$. This integral has been studied for the case $n=3$, by Kobb;* and in the general case by Radon.† It is found that a minimizing surface must satisfy the partial differential equations

$$P_i \equiv F_{x_i} - \sum_1^{n-1} \frac{\partial}{\partial u_\mu} F_{x_{i\mu}} = 0,$$

which are not independent, but are connected by the $n-1$ relations $\sum_i P_i x_{i\mu}=0$.

The first question, which arises in the study of the problem in this form, refers to the conditions under which the integral (28) shall be independent of the choice of the parameters $u_1, \cdots, u_{n-1}$. The following treatment of this question is due to Radon. As in the case of the simple problem,‡ we find that if the integral (28) is to be invariant under a transformation of parameters $u_\mu = u_\mu(u_1', \cdots, u_{n-1}')$ which preserves the orientation upon a surface, i.e., such that

$$\Delta = \partial(u_1, \cdots, u_{n-1})/\partial(u_1', \cdots, u_{n-1}') > 0,$$

then we must have

$$F(x_i, x_{i\mu}') = \Delta F(x_i, x_{i\mu}). \tag{29}$$

Radon observes that if p_h represents the determinant obtained from the matrix $||x_{i\mu}||$ by omitting the hth row and prefixing the factor $(-1)^h$, then condition (29) is

* See ACTA MATHEMATICA, vol. 16 (1892), p. 65.

† See MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 22 (1911), p. 53.

‡ See I, §4(c).

equivalent to the condition that $F(x_i, x_{i\mu})$ can be expressed in terms of x_i and p_i, $F(x_i, x_{i\mu}) = \Phi(x_i, p_i)$, and that for $k > 0$, we shall have

$$\Phi(x_i, kp_i) = k\Phi(x_i, p_i) ;$$

a condition comparable in simplicity with the homogeneity condition (14) for the simple integral in parametric form. From this relation we obtain by differentiating with respect to k and then putting $k=1$, the equation $\sum_i p_i \Phi_{p_i} = \Phi$, and hence by differentiating with respect to p_j we find the system of equations $\sum_j p_j \Phi_{p_i p_j} = 0$. From this we conclude that there must exist $n(n-1)/2$ functions Φ_{rs} such that

$$\Phi_{p_i p_j} = \sum_{rs} x_{ir} x_{js} \Phi_{rs}, \qquad \Phi_{rs} = \Phi_{sr} ;$$

these functions Φ_{rs} play a role analogous to that of the function F_1 in the simple problem. Instead of to a set of n partial differential equations between which there exist $n-1$ linear relations, the first order condition now leads to a single equation which a solution of the problem must satisfy, viz.,

$$T \equiv \sum_{i=1}^{n} \Phi_{x_i p_i} + \sum_{r,s=1}^{n-1} \sum_{h=1}^{n} \Phi_{rs} x_{hs} p_{hr} = 0,$$

an equation which resembles in form the Weierstrass equation for the simple problem

$$F_{xy'} - F_{x'y} + F_1(x'y'' - x''y') = 0.$$

Radon also obtains a simple expression for the transversality condition, a generalization of the Hilbert invariant integral, and of the Weierstrass formula.

The invariance of the integral (28) under parameter transformation has also been studied by Vivanti,* by Usai,†

* See Palermo Rendiconti, vol. 33 (1912), p. 268; Annali di Matematica, vol. 20 (1913), p. 49; Palermo Rendiconti, vol. 47 (1922), p. 232.

† See Giornale di Matematiche, vol. 52 (1914), p. 63; vol. 53 (1915), p. 136; Lombardo Rendiconti, vol. 48 (1915), p. 77; ibid., vols. 49, 52; Annali di Matematica, vol. 31 (1922), p. 279.

and by Grosz.* Vivanti's earlier papers followed the method of Radon, giving, however, details of the reasoning which led to the conclusion that $F(x_i, x_{i\mu})$ must be expressible in the form $\Phi(x_i, p_i)$. Usai had attacked the problem by means of a partial differential equation for the function F and had considered also the case in which F contained partial derivatives of the x_i of the second, third, and fourth orders. This was highly complicated work involving very elaborate calculations. In his 1922 paper, Vivanti developed a very simple method leading to a condition for the invariance of the simple integral $\int F(x, y, x', y', x'', y'', \cdots, x^{(n)}, y^{(n)})dt$, which may be stated as follows.

Let $\psi_1(t) = y'/x'$, and let $\psi_r(t)$ be defined by the recursion formula $\psi_r(t) = x'\psi_{r-1}(t) - (2r-3)x''\psi_{r-1}(t)$; these functions ψ_r are the numerators of the formulas that express d^ry/dx^r in terms of the derivatives with respect to t of x and y. Then the necessary and sufficient condition for the invariance of the integral is that F be a positively (or negatively) homogeneous function of degree 1 with respect to $x', y', [\psi_2(t)]^{1/3}, \cdots, [\psi_r(t)]^{1/(2r-1)}, \cdots, [\psi_n(t)]^{1/(2n-1)}$. After the appearance of the paper, Usai, who had obtained an equivalent result by his elaborate methods, returned to the problem of the invariance under parameter transformation of an n-fold integral involving partial derivatives of higher order, viz.,

$$\int_{(n-1)} F(x_i, x_{i\mu_1}, x_{i\mu_1\mu_2}, \cdots, x_{i\mu_1\cdots\mu_k})du_\mu ,$$

for $\mu, \mu_1, \cdots, \mu_k = 1, \cdots, n-1$; and obtained a result of the same character as that of Vivanti. If x_n be looked upon as a function of the remaining $n-1$ variables $x_1, \cdots, x_{n-1}$, we find that $\partial x_n/\partial x_\mu = -p_\mu/p_n$, where p_μ and p_n are the functions defined by Radon. Similarly, it is found that

* See Monatshefte für Mathematik und Physik, vol. 27 (1916), p. 70.

$$\frac{\partial^2 x_n}{\partial x_{\mu_1} d x_{\mu_2}} = \frac{\sigma_{\mu_1\mu_2}}{p_n^3}, \quad \cdots, \quad \frac{\partial^k x_n}{\partial x_{\mu_1} \cdots \partial x_{\mu_k}} = \frac{\sigma_{\mu_1\mu_2\cdots\mu_k}}{p_n^{2k-1}},$$

where $\sigma_{\mu_1\mu_2}, \cdots, \sigma_{\mu_1\mu_2\ldots\mu_k}$ are polynomials in the partial derivatives of x_i up to those of order 2, $\cdots$, k respectively. The condition for invariance is now that F be expressible as a function of x_i, p_i, $(\sigma_{\mu_1\mu_2})^{1/3}, \cdots, (\sigma_{\mu_1\mu_2\ldots\mu_k})^{1/(2k-1)}$ and that it be positively (or negatively) homogeneous of degree 1 with respect to p_i, $(\sigma_{\mu_1\mu_2})^{1/3}, \cdots, (\sigma_{\mu_1\mu_2\ldots\mu_k})^{1/(2k-1)}$.

These results answered definitively the question of the invariance of an $(n-1)$-fold integral involving n functions of $n-1$ parameters. The question still remained open with regard to an m-fold integral, when $1<m<n-1$. This was answered, at least for integrals involving first partial derivatives, by Grosz in the paper referred to above. He considers the integral

$$\int_{(n)} F(z_k, z_{ki}) du_1 \cdots du_n,$$

for $k=1, \cdots, N=n+m$, and $i=1, \cdots, n$. Putting also $p_0=\partial(z_1, \cdots, z_n)/\partial(u_1, \cdots, u_n)$, and defining the symbol $p_{ij}(i=1, \cdots, n;\ j=1, \cdots, N-n)$ as the determinant obtained from p_0 by omitting z_i and placing z_{n+j} behind z_n, he finds that F must be positively homogeneous of degree 1 with respect to p_0 and p_{ij} at any point at which the matrix of the z_{ki} is of rank n.

2. *Euler-Lagrange Rule*; *Jacobi's Condition.* The last mentioned paper offers a convenient transition to other parts of the theory of the multiple integral.

(a) The paper by Grosz is concerned primarily with an extension of the Lagrange problem to the double integral. The problems considered are variations on the following central theme of the paper.* To minimize the double integral $\int_B \int f(z_k; z_{kr})\, du_1 du_2$, $k=1, \cdots, n+2$; $r=1, 2$, when z_k are to be functions of class C'' which satisfy the finite

* See the second footnote on p. 491.

equations $g_\pi(z_1, \cdots, z_{n+2})=0$, $\pi=1, \cdots, p<n$, and the isoperimetric conditions $\int_B\int f_\gamma(z_k;z_{kr})du_1du_2=C_\gamma$, $\gamma=1, \cdots, q$, and which take given values on the boundary of the domain B over which the integration takes place. To insure non-singularity the requirement is made that the matrix $\|z_{kr}\|$ is everywhere of rank 2. For this problem, an extension of the multiplier rule is obtained by the methods developed by Hilbert and Bolza* for the classical Lagrange problem. The result may be stated as follows. If $z_k=Z_k(u_1,u_2)$ is a solution of the problem, then there must exist $q+1$ constants $l_0, \cdots, l_q$ and p functions $\lambda_\pi(u_1,u_2)$ such that Z_k and λ_π satisfy the partial differential equations

$$F_{z_k} - \frac{\partial}{\partial u_1}F_{z_{k1}} - \frac{\partial}{\partial u_2}F_{z_{k2}} = 0,$$

where $F\equiv l_0f+\sum l_\gamma f_\gamma+\sum\lambda_\pi g_\pi$.

The variations on this theme are obtained by modifications of the conditions which the functions z_k have to satisfy on the boundary of the domain of integration B, and by the consideration of various possibilities for this boundary, of which only rectifiability is presupposed.

In spite of the considerable generality that is thus attained with respect to some elements of the problem, we are still far from having a theory of the general Lagrange problem for double integrals. Grosz goes further however and obtains an extension of the multiplier rule for the problem in which the integral $\int_B\int f(x,y,z,u,z_x,z_y,u_x,u_y)dxdy$ is to be minimized by functions which must also satisfy the partial differential equation $z_x=g(x, y, z, u, u_x)$.

(b) An interesting group of investigations is concerned with the analog of Jacobi's condition in the theory of the integral $\int_B\int f(x, y, z, p, q)dxdy$, where $p\equiv\partial z/\partial x$ and $q\equiv\partial z/\partial y$. As usually stated in the literature, this condition is that for no simple closed analytic curve C_1 which lies

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* See the first footnote on p. 491.

entirely within B, shall there exist a function $\zeta(x,y)$, except $\zeta\equiv 0$, which satisfies the equation

$$\frac{\partial}{\partial x}(f_{pp}\zeta_x + f_{pq}\zeta_y) + \frac{\partial}{\partial y}(f_{qp}\zeta_x + f_{qq}\zeta_y) + \zeta\left(\frac{\partial}{\partial x}f_{zp} + \frac{\partial}{\partial y}f_{zq} - f_{zz}\right) = 0,$$

which is continuous on C_1 and of class C'' in the interior of C_1, and which vanishes on C_1. Because this condition is not readily applicable, it has been a desideratum to replace this condition by another in more usable form. One way in which this has been done recently* connects up most readily with the Jacobi condition III for the simplest problem. Writing $f_{y'y'}\equiv R$ and $f_{yy}-df_{yy'}/dx\equiv -A$, the Jacobi equation (8) takes the form $(Ru')'+Au=0$. Consider now the boundary problem consisting of the differential equation

$$(Ru')' + \lambda A u = 0, \tag{30}$$

and the conditions $u(x_1)=0$, $u'(x_1)=1$. For this problem there exists an infinite set of positive characteristic constants $\{\lambda_n,\ n=0,\ 1,\ \cdots\}$ and a corresponding system of functions $\{u_n(x)\}$, which satisfy the conditions of the problem and the further condition $u_n(x_2)=0$. It has been shown by Picone† that a necessary and sufficient condition for the existence of a solution of the problem which vanishes exactly n times on $(x_1,\ x_2)$ is that $\lambda_{n-1}\leqq\lambda<\lambda_n$, it being understood that we write 0 in place of λ_{-1}. Hence if $\lambda_0>1$, there exists no solution of Jacobi's equation which vanishes between x_1 and x_2; and if $\lambda_0\leqq 1$, there exists such a solution of the equation. This result makes it possible to replace Jacobi's condition by the condition that the least positive

* See Picone, Rendiconti dei Lincei, vol. 30 (1921), p. 410; also vol. 31 (1922), p. 46 and p. 94.

† See Annali Scuola Normale di Pisa, vol. 11 (1909), p. 3.

characteristic number λ_0 for the boundary problem (30) be greater than 1.

The proof of the necessity of the condition $\lambda_0 \geqq 1$ for a permanent sign of the second variation is simply made. For, λ_0 being less than 1, let us suppose that $\lambda_{\nu-1} < 1 \leqq \lambda_\nu$. Multiplying the equation $(Ru_n')' + \lambda_n A u_n = 0$ through by u_n, and integrating, we find

$$-\lambda_n \int_{x_1}^{x_2} A u_n^2 dx = \int_{x_1}^{x_2} u_n (Ru_n')' dx$$

$$= u_n u_n' R \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} R u_n'^2 dx,$$

or

$$\int_{x_1}^{x_2} R u_n'^2 dx = \lambda_n \int_{x_1}^{x_2} A u_n^2 dx.$$

Hence, if Legendre's condition for a minimum is satisfied, $\int_{x_1}^{x_2} A u_n^2 dx > 0$; and moreover, we can show that for $n < \nu$, $\int_{x_1}^{x_2} (R u_n'^2 - A u_n^2) dx < 0$, while $\int_{x_1}^{x_2} (R u_n'^2 - A u_n^2) dx \geqq 0$ for $n \geqq \nu$. From this we conclude that if Jacobi's condition is not satisfied, the second variation can be made negative as well as positive by functions u which vanish at x_1 and x_2.

The boundary problem (30) is equivalent to the linear integral equation*

$$u(x) = \lambda \int_{x_1}^{x_2} G(x, \xi) A(\xi) u(\xi) d\xi,$$

in which $G(x, \xi)$ is the corresponding Green's function; and the characteristic values λ_n are the roots of the Fredholm determinant for this equation, $F(\lambda)$. We are therefore able to replace Jacobi's condition by the condition that $F(\lambda) \neq 0$ on (0, 1). It is this last form of Jacobi's condition which Picone has carried over to the theory of the double integral.

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* See, e.g., Bôcher, *Leçons sur les Méthodes de Sturm*, p. 108 et seq.

The results which have here been indicated, and which run very close to investigations of Richardson,* had previously been given by Lichtenstein in papers† which carry the problem a great deal farther. Starting from the work of Schwarz on minimal surfaces,‡ he follows, for the general double integral, the method suggested above, and discusses also the case in which the least positive characteristic constant λ_0 is equal to 1. He proved also that if $\lambda_0>1$, the extremal can be imbedded in a field, and then extends the results to include the case in which the boundary of the minimizing surface is not fixed.

This latter case forms the subject of a recent doctoral dissertation by Simmons.§ He secured a very simple form for the second variation of the integral and deduced from this a formulation of the Jacobi condition in terms of a boundary value problem.

(c) Mention should finally be made of another method of dealing with the second variation of the double integral in parametric form, the object of which is to make a reduction of the second variation for this case to the form which it has in the non-parametric case. This reduction is based on a treatment of linear partial differential forms, similar to that used by the author for ordinary linear differential forms in the study of the second variation of the simple integral in parametric form.‖

3. *A General Formulation.* The discussion of the problems which have been mentioned thus far makes it possible to

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* See MATHEMATISCHE ANNALEN, vol. 68 (1910), p. 279; vol. 71 (1911), p. 214; TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), p. 22; vol. 18 (1917), p. 489.

† See SITZUNGSBERICHTE DER BERLINER GESELLSCHAFT, vol. 14 (1915), p. 119; MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 28 (1917), p. 1; MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), p. 26.

‡ See GESAMMELTE ABHANDLUNGEN, vol. 1, p. 223.

§ See Report of the April meeting of this Society, this BULLETIN, vol. 32 (1926), p. 221.

‖ See ANNALS OF MATHEMATICS, (2), vol. 13 (1912), p. 149; vol. 15 (1913), p. 78.

formulate now in general terms the problems of the calculus of variations involving integrals of various kinds. In every such problem we are concerned with an integral of the following form:

$$\int_M F\left(t_1, \cdots, t_m ; x_1, \cdots, x_n ; \frac{\partial x_\nu}{\partial t_\mu} ; \cdots ; \frac{\partial^r x_\nu}{\prod_{\rho=1}^{r} dt_{\mu_\rho}}\right) dt_1 \cdots dt_m,$$

in which $m \leqq n$, and r is an arbitrary integer, and in which the function F is subject to conditions of continuity. The integration is extended over a closed domain M in the m-dimensional space of the variables t_μ, which may be completely determined, or which may merely be conditioned by restrictions that determine it incompletely.

In the second place we are concerned with a class K_0 of sets of functions $x_1(t_\mu), \cdots, x_n (t_\mu)$, or with a class $\overline{K}_0$ of m-dimensional manifolds in n-dimensional space $(m<n)$, represented by a set of functions $x_\nu(t_\mu)$ or by a set of functions $X_\nu(\tau_\mu)$, where

$$X_\nu(\tau_\mu) = x_\nu[T_1(\tau_1, \cdots, \tau_m), T_2(\tau_1, \cdots, \tau_m), \cdots, T_m(\tau_1, \cdots, \tau_m)],$$

in which the functions $T_\mu(\tau_1, \cdots, \tau_m)$ are of class C' and such that $\partial(T_1, \cdots, T_m)/\partial(\tau_1, \cdots, \tau_m) > 0$. The class K_0 is further restricted by conditions on the functions which make up the set $x_1, \cdots, x_n$; these conditions may be in the first place continuity conditions, and further they may take the form of finite equations, or of differential equations, ordinary or partial; finally they include conditions which these functions must satisfy on the boundary of the closed domain M over which the integration is extended.

Finally we are concerned with a subclass K_1 of the class K_0; this subclass usually is defined by means of such sets of functions $x_1, \cdots, x_n$ as lie within a prescribed neighborhood of a particular one of these sets. This neighborhood may be of order 0, 1, $\cdots$, p, according as the conditions defining it include 1, 2, $\cdots$, $p+1$ of the inequalities

$$| y_\mu - x_\mu | < \rho, \quad | y_\mu' - x_\mu' | < \rho, \quad \cdots, \quad | y_\mu^{(p)} - x_\mu^{(p)} | < \rho.$$

With these elements in mind, we may now state the general problem to be one in which we ask for an element of class K_0, which shall give to the integral a value not greater (less) than that which it has for any element of class K_1. If $K_1 \equiv K_0$, we have the problem of an absolute minimum (maximum); if K_1 is a proper subclass of K_0 we have a problem of a relative minimum (maximum), the order of the problem being determined by the type of neighborhood involved in the definition of K_1.

IV. Absolute Minima and Functional Calculus

1. *Minima of Functionals.* In 1904, Hilbert formulated the following general minimum problem. An infinite class K of mathematical objects a, b, $\cdots$ is given; also an operator J associating with each element a real number. To determine an element of K to which corresponds the smallest number.* Apart from the evident vagueness of this problem, it is clear that it includes a vast number of problems many of which are not, as far as we can see now, problems of the calculus of variations and are not reducible to such problems. Among them are the ordinary maxima and minima problems, problems like that of Kakeya† which asks for the least area in which it is possible to turn a line of given length through an angle of 180° and also the type of problem which has been discussed by Bonnesen,‡ Lebesgue,§ Blaschke|| and others. It includes the general problem of the minima and maxima of functionals. And the

* See Bolza, *Vorlesungen*, p. 16, footnote.

† See, e.g., Ford, this Bulletin, vol. 28 (1922), p. 45.

‡ See Mathematische Annalen, vol. 91 (1924), p. 252; vol. 95 (1925), p. 267.

§ See Journal de Mathématiques, (8), vol. 4 (1921), p. 67.

|| See Mathematische Zeitschrift, vol. 6 (1920), p. 281.

calculus of variations may indeed be conceived as a part of the broader theory.

This standpoint, clearly suggested by the work of Volterra, was definitely introduced in the calculus of variations by Hadamard,* but only in a tentative way. It has been systematically developed by Tonelli, in his *Fundamenti di Calcolo delle Variazioni*, to which the greater part of the present section will be devoted. In the discussion of the Euler-Lagrange multiplier rule we have already called attention to the extension of this rule by Hahn† to the case in which the integrals are replaced by more general functionals. Mention must also be made of a paper by LeStourgeon,‡ which is concerned with minima of functionals (in particular of functions of lines), of such character as to include the integrals of the calculus of variations. The independent variables are arcs of curve defined by equations

$$y = \lambda(x), \qquad a \leqq x \leqq b.$$

For each arc λ of a class L of such arcs there is defined a real number $F(\lambda)$. The problem considered is that of determining an arc λ_0 of L for which there shall exist a number d, of such character that for every arc of L for which $|\lambda-\lambda_0| < d$, $|\lambda'-\lambda_0'| < d$, we shall have $F(\lambda) \geqq F(\lambda_0)$. We say that $F(\lambda)$ has a differential at λ_0 if there exists a linear functional $L(\eta)$ with continuity of order 1, such that

$$F(\lambda_0+\eta) - F(\lambda_0) = L(\eta) + \epsilon(\eta) M_1(\eta),$$

where $M_1(\eta)$ is the maximum $|\eta|$ and $|\eta'|$ on (a, b), and $\epsilon(\eta) \to 0$ with $M_1(\eta)$. Similarly, $F(\lambda)$ is said to have a second differential at λ_0 if there exists a linear functional $L(\lambda)$ and a bilinear functional $B(\lambda, \mu)$, such that

$$F(\lambda_0 + \eta) - F(\lambda_0) = L(\eta) + B(\eta, \eta) + M^2(\eta) \cdot \epsilon(\eta),$$

where $\epsilon(\eta)$ and $M(\eta)$ are defined as before.

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* See *Leçons sur le Calcul des Variations*, Book 2, Chap. 7.

† See II, §2(c).

‡ See Transactions of this Society, vol. 21 (1920), p. 357.

By a modification of the theory of Riesz, referred to above,* it is shown that a linear functional $L(\lambda)$ which has continuity of order 1, is expressible, in an infinitude of ways, in the form

$$L(\lambda) = \int_a^b \lambda(x)du(x) + \int_a^b \lambda'(x)du_1(x),$$

where $u(x)$ and $u_1(x)$ are functions of limited variation on (a, b); and also that a bilinear functional $B(\lambda, \mu)$ which has continuity of order 1 in each variable, when the other is fixed, can be expressed in the form

$$B(\lambda,\mu) = \int_a^b\int_c^d \lambda(x)\mu(y)d_{xy}p(x,y) + \int_a^b\int_c^d \lambda'(x)\mu(y)d_{xy}q'(x,y)$$
$$+ \int_a^b\int_c^d \lambda\mu' d_{xy}q''(x,y) + \int_a^b\int_c^d \lambda'\mu' d_{xy}r(x,y)$$

in which p, q', q'', r are of limited variation in x and y separately and jointly.

It is now shown that conditions of the first and second orders for a minimum of $F(\lambda)$, which is supposed to possess a second differential, are obtained from the conditions $L(\eta)=0$ and $B(\eta, \eta)\geqq 0$; and from these, generalizations of the Euler equation, of the transversality condition and of the Jacobi condition are obtained, the latter by applying Bliss' method† to the condition $B(\eta, \eta)\geqq 0$.

2. *Introduction to Tonelli's Work.* In the two volumes of Tonelli's *Fondamenti* which have thus far appeared, the simplest problem of the calculus of variations is discussed, both in the form (1) and in the parametric form (13), with and without an isoperimetric condition. We shall confine our brief account of this work to the theory of the unrestricted integral (13). This integral associates a real number with every curve C of a certain class, and is therefore a function of such curves; the integral is studied with regard to its dependence upon the curve.

* See the footnote on p. 496.

† See I, §5 (b).

The various problems of the calculus of variations which have been discussed in earlier sections, are concerned with the relative extremes of an integral; i.e., they are problems for which the class K_1 is a proper subclass of K_0. To secure knowledge of the absolute extremes, one might conceivably, after having determined all relative extremes, determine that one among them, which gives to the integral the smallest value.

But this method leaves out of account the difficulty that arises in case there is an infinitude of relative extremands, in which case one cannot be sure that any absolute extreme exists. The question as to the existence of an absolute extreme has given rise to the famous Dirichlet principle, according to which it was argued that

$$\int\int (u_x^2 + u_y^2)\, dxdy$$

must have a minimum, because its value is always $\geqq 0$. That this conclusion involves a confusion between minimum and greatest lower bound was pointed out by Weierstrass.

A first step in the direction of establishing conditions which will insure the existence of an absolute extreme was taken by Hilbert,* who proved an existence theorem, which has since been extended to the following form.

If (1) $F(x, y, x', y')$ is of class C''' and positively homogeneous of degree 1 in x' and y' in a region T, defined by (x, y) in R, $x'^2+y'^2 \neq 0$,

(2) $F(x, y \cos \gamma, \sin \gamma) > 0$ for (x, y) in R_0, a domain in the interior of R, and γ arbitrary,

(3) F_1 $(x, y, \cos \gamma, \sin \gamma) > 0$, for (x, y) in R_0 and γ arbitrary,

(4) R_0 is bounded, closed, and convex,

(5) A_1 and A_2 are two distinct points of R_0.

* See Jahresbericht der Vereinigung, vol. 8 (1899), p. 184; see also Bolza, loc. cit., Chap. IX.

Then there exists at least one rectifiable curve H joining A_1 and A_2, which furnishes an absolute minimum for a suitably generalized integral

$$\int F(x,y,x',y')dt,$$

with respect to the totality of rectifiable curves which join A_1 and A_2 and which lie in R_0. A large part of Tonelli's second volume is devoted to existence theorems of this general character. That we have a problem here very different from the corresponding one in the theory of functions of a real variable becomes clear when we realize that the functionals of the calculus of variations do not possess the continuity property which is basal in the other theory. A very wide class of integrals of the calculus of variations possess lower semicontinuity, which property accordingly occupies a central position in Tonelli's treatment.* In order to obtain an idea of the scope of this treatment, we must take up a few agreements and definitions. The integral (13) taken along a curve C is denoted by I_C.

(1) A point set A of the xy-plane, such that those of its points which belong to any circle form a *closed* set, is called a region. The function $F(x, y, x', y')$ is understood to be of class C, and of class C'' with respect to x' and y' for (x, y) in A and $x'^2+y'^2\neq 0$; and also to satisfy the homogeneity condition (14).

(2) If $F_1>0(<0)$ for (x, y) in A and $x'^2 + y'^2\neq 0$, the integral I_C is called *positively* (*negatively*) *regular*; if $F_1\geqq 0(\leqq 0)$, I_C is called *positively* (*negatively*) *quasi-regular.*

(3) If $F_1\geqq 0$ for (x, y) in A and $x'^2+y'^2\neq 0$, and if for *no point* (x, y) *in* A, the values of θ for which $F_1(x, y, \cos\theta, \sin\theta)=0$ fill up any subinterval of $(0, 2\pi)$, I_C is called *normally quasi-regular.*

(4) If $F_1\geqq 0$ for (x, y) in A and $x'^2+y'^2\neq 0$, and if for *no point* in A, we have $F_1(x, y, \cos\theta, \sin\theta)=0$ for all θ, I_C is called *semi-normally quasi-regular.*

* See an article by Tonelli, this BULLETIN, vol. 31 (1925), p. 163.

(5) If $F>0(<0)$ for (x, y) in A and $x'^2+y'^2\neq 0$, I_C is called *positively* (*negatively*) *definite*; if $F\geqq 0(\leqq 0)$, I_C is called *positively* (*negatively*) *semi-definite.*

(6) An *ordinary* curve is one which lies in the region A and which is rectifiable.

(7) A class K of ordinary curves is called *complete* if every *rectifiable accumulation curve* of the set also belongs to the set.

3. *Tonelli's Treatment.* With these definitions in mind, we proceed to the following theorem.

(a) If I_C is positively quasi-regular, and if for a given complete class K of ordinary curves C, all in a bounded portion A_1 of A, it is possible to determine a function $\Phi(\alpha)$, defined and continuous for all real values of α, always non-negative and non-decreasing, such that for any curve C of K, we have

$$L\leqq\Phi(I_C),$$

in which L is the length of C, then I_C has an absolute minimum in K.

We shall indicate the principal steps in the proof of this theorem, so as to obtain an insight into Tonelli's method and because it acquaints us with some of the central features of his theory. Let $\{C_n\}$, $n=1, 2, \cdots$, be a sequence of sets of ordinary curves taken from K, such that for every curve C_n of the set $\{C_n\}$, we shall have

$$I_{C_n} < i+\frac{1}{n}, \quad \text{or} \quad I_{C_n} < -n,$$

i being the greatest lower bound of the values of I_C in K, according as i is finite or infinite, this greatest lower bound being necessarily equal to $-\infty$ in the latter case. We show then first of all that i is finite; for in the contrary case we would have in virtue of the hypotheses on $\Phi(\alpha)$, if L_n denotes the length of C_n,

$$L_n \leqq \Phi(I_{C_n}) \leqq \Phi(-n) \leqq \Phi(-1).$$

Hence, if M is the maximum of $|F(x, y, \cos\theta, \sin\theta)|$ for (x, y) in A_1 and for $0 \leqq \theta \leqq 2\pi$, we shall have

$$I_{C_n} > -ML_n \geqq -M\Phi(-1);$$

but this result contradicts the condition $I_{C_n} < -n$.

We conclude that we have for every curve C_n of the set $\{C_n\}$, $I_{C_n} < i + 1/n$; and consequently

$$L_n \leqq \Phi(I_{C_n}) \leqq \Phi\left(i + \frac{1}{n}\right) \leqq \Phi(i+1),$$

so that the lengths of the curves of the sets $\{C_n\}$ are bounded. Now we make use of an important theorem of Hilbert* according to which an "infinite set of continuous curves, all contained in a bounded region and whose lengths form a bounded set, has at least one continuous and rectifiable accumulation curve." This theorem has played an important part in Hilbert's existence proofs, and it is not surprising to see it appear here. We conclude from it that the set of curves K has a rectifiable accumulation curve C_0. It is readily proved that C_0 must lie within the region A and that its length L_0 is less than or equal to $\Phi(i+1)$; hence, since the class K is complete, this curve C_0 belongs to K.

Now Tonelli has proved† that "If I_C is positively quasi-regular, and L is an arbitrary positive number, then I_C is a lower semi-continuous function on the set of ordinary curves whose length is less than L." From this it follows that I_C is lower semi-continuous on C_0, which means that for any ϵ, there exists a ρ, such that for any ordinary curve C which lies in the ρ-neighborhood of C_0, we have

$$I_{C_0} < I_C + \epsilon .$$

In particular, therefore, there exists an n_ϵ, such that if $n > n_\epsilon$, there is at least one curve C_n for which

$$I_{C_0} < I_{C_n} + \epsilon .$$

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* See JAHRESBERICHT DER VEREINIGUNG, vol. 8 (1900), p. 184.

† See *Fondamenti*, vol. I, p. 292.

Consequently, $I_{C_0}<i+1/n+\epsilon$, for every $n>n_\epsilon$; hence we conclude $I_{C_0}\leqq i+\epsilon$, and therefore $I_{C_0}\leqq i$. On the other hand, since C_0 is in K, we must have $I_{C_0}\geqq i$; therefore we obtain the final result that $I_{C_0}=i$, which means that the rectifiable curve C_0 furnishes an absolute minimum for I_C.

(b) There is a second criterion for the existence of an absolute minimum of I_C, for the case that I_C is positively quasi-regular and positively semi-definite; the proof depends in an essential manner on the fact that also in this case I_C is lower semi-continuous. From these two criteria, twelve existence theorems for an absolute minimum are derived which form a nucleus for the further developments and which constitute therefore a very fundamental part of the work. Thus it is seen how essential a role the lower semi-continuity of I_C plays in Tonelli's theory of the absolute minimum, and it is not surprising therefore that the first volume of his work has as its central purpose the study of this property. There are given a large number of necessary conditions, of sufficient conditions, and of conditions which are both necessary and sufficient for the lower semi-continuity of I_C, over the entire class of ordinary curves (i. e. uniform lower semi-continuity), over a restricted class of curves, or on a particular curve. Among the results that are obtained it is interesting to note the following:

(1) If I_C is positively quasi-regular and positively definite, it is lower semi-continuous.

(2) If I_C is lower semi-continuous in A, then $F_1(x, y, \cos\gamma, \sin\gamma)\geqq 0$ for every γ and for every point (x, y) which is interior to A or is a limiting point of interior points of A.

(3) If A satisfies a certain condition of convexity, and if I_C is positively definite, then a necessary and sufficient condition for the lower semi-continuity of I_C on an ordinary curve C_0 in A is that for "nearly every point" of C_0 and for all values of θ, we have

$$E(x_0, y_0, \cos\theta_0, \sin\theta_0, \cos\theta, \sin\theta)\geqq 0 .$$

4. *Further Developments.* Once the question as to the existence of a curve which furnishes an absolute minimum has been answered more or less completely, attention is turned to the properties of such curves.

(a) From the conditions for semi-continuity and the relation of this property to the existence of minima, the necessity of the conditions of Legendre and Weierstrass follows without difficulty. It is next shown in the usual way that at every point of a minimizing curve of class C' which lies in the interior of the region A, the Euler equations must be satisfied; also, that every ordinary extremand which is not of class C' must satisfy the equations

$$\int_0^s F_x\,ds - \frac{d}{ds}\int_0^s F_{x'}\,ds = C_1, \quad \int_0^s F_y\,ds - \frac{d}{ds}\int_0^s F_{y'}\,ds = C_2,$$

along every one of its arcs whose points fall entirely within A; a curve satisfying these equations is called an extremaloid.

(b) When we come to the question of the existence of extremals and extremaloids, it is clear that this can be answered by means of the existence theorems for absolute minima. For a variety of conditions are known under which there must exist at least one minimizing curve; moreover, such a curve must be an extremal or an extremaloid—hence we can infer the existence of extremals (extremaloids).

We observe that this procedure suggests a new method for proving the existence of solutions of differential equations of the second order, since such an equation may always be looked upon as the Euler equation of some problem of the calculus of variations. At any rate the existence of extremals is put on a basis independent of the existence theorems for differential equations. In consequence of this the problems concerning the uniqueness of extremals, their dependence on the initial points and their differentialibity with respect to initial data receive a treatment which does not, as usually has been the case, depend upon the theory of differential equations. And finally, necessary and sufficient conditions for relative minima are derivable from those for absolute minima; thus a connection is established with the classical theory.

(c) For the simplest problems of the calculus of variations, Tonelli's method dominated by the point of view of the functional calculus, has led to an important re-orientation of the subject, and to the establishment of an imposing group of existence theorems. How fruitful it will prove to be in the study of more complicated problems, and whether it will enable us to study functionals of more general character than those included in the calculus of variations, are questions which I am not able to answer. Certainly, our knowledge concerning the minima of multiple integrals, at present handicapped by the incompleteness of our knowledge concerning partial differential equations, would be greatly enlarged if the functional methods were to prove extensible so as also to cover these cases.

V. Appendix

In this report on progress in the calculus of variations, I have of necessity limited myself to a few of the subdivisions of this field. No mention has been made, e. g., of the applications and extensions which are being made in the newer differential geometry of calculus of variations concepts. Moreover, I could not aspire to completeness in those parts of the subject with which I did deal. While very useful bibliographies for the calculus of variations up to 1920 have been published by Lecat,* it will be worth while to list here, although merely by title, some additional papers which have come to my attention, and which are not mentioned by Lecat, most of them being subsequent to 1920.

* See *Bibliographie du Calcul des Variations, depuis les origines jusqu' à 1850*, Paris, Hermann; *Bibliographie du Calcul des Variations*, 1850–1913, Paris, Hermann; Appendix to the *Bibliographie des Séries Trigonométriques*, published by the author, Avenue des Alliés, 92, Louvain, which carries the bibliography of the calculus of variations up to 1920.

BIBLIOGRAPHY

I. SIMPLEST PROBLEM.

Carathéodory, *Sur les points singuliers du problème du calcul des variations dans le plan*, ANNALI DI MATEMATICA, (3), vol. 21 (1913), p. 153.

Carathéodory, *Sui campi di estremali uscenti da un punto e riempienti tutto lo spazio*, BOLLETTINO DELLA UNIONE MATEMATICA ITALIANA, vol. 2 (1923), pp. 2, 48, 81.

Hammerstein, *Elementarer Beitrag zur Variationsrechnung*, MATHEMATISCHE ANNALEN, vol. 87 (1922), p. 228.

Lichtenstein, *Über einige Existenzprobleme der Variationsrechnung*, JOURNAL FÜR MATHEMATIK, vol. 145 (1914), p. 24.

Pchéborsky, *Note sur le calcul des variations* (in Russian), COMMUNICATIONS DE LA SOCIÉTÉ DE KHARKOFF, (2), vol. 14 (1915), p. 224.

Pchéborsky, *Problems concerning the extreme of the integral* $\int f(x,y,y')dx$ *with variable end-points* (in Russian), ANNALES DES INSTITUTIONS DE L' UKRAINE, Section Mathématique, vol. 1 (1924), p. 1.

Picone, *Sul calcolo delle variazioni*, PALERMO RENDICONTI, vol. 46 (1922), p. 437.

Razmadze, *Sur une condition de minimum nécessaire pour les solutions anguleuses dans le calcul des variations*, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 51 (1923), p. 223.

II. THE LAGRANGE PROBLEM.

Hadamard, *La construction de Weierstrass et l' existence de l' extremum dans le problème isopérimétrique*, ANNALI DI MATEMATICA, vol. 21 (1913), p. 251.

Miles, *Some properties of space curves minimizing a definite integral with discontinuous integrand*, this BULLETIN, vol. 20 (1913), p. 11.

Rider, *On the* F_1 *function of the calculus of variations*, WASHINGTON UNIVERSITY STUDIES, vol. 5 (1918), p. 97.

Rider, *On the minimizing of a class of definite integrals*, ANNALS OF MATHEMATICS, (2), vol. 24 (1922), p. 167.

Taylor, *Reduction of Euler's equation to a canonical form*, this BULLETIN, vol. 31 (1925), p. 259.

III. MULTIPLE INTEGRALS.

Boehm, *Der Unabhängigkeitssatz für Doppelintegrale*, MATHEMATISCHE ANNALEN, vol. 83 (1921), p. 149.

Boehm, *Über eine Eigenschaft der Minimalflächen*, MATHEMATISCHE ANNALEN, vol. 83 (1921), p. 157.

Carathéodory, *Sur le traitement géométrique des extrèmes des intégrales doubles*, ENSEIGNEMENT MATHÉMATIQUE, vol. 19 (1917), p. 329.

Carathéodory, *Über die kanonischen Veränderlichen in der Variationsrechnung der mehrfachen Integrale*, MATHEMATISCHE ANNALEN, vol. 85 (1922), p. 78.

Carathéodory, *Über ein Reziprozitätsgesetz der verallgemeinerten Legendreschen Transformation*, MATHEMATISCHE ANNALEN, vol. 86 (1922), p. 272.

Koschmieder, *Invarianten bei der Variation vielfacher Integrale*, MATHEMATISCHE ZEITSCHRIFT, vol. 24(1925), p. 181.

Kürschak, *Über das identische Verschwinden der Variation*, ACTA LITERARUM AC SCIENTIARIUM R.U.H. FRANCIS-JOSEPHINAE, vol. 1 (1922), p. 6.

IV. MISCELLANEOUS TOPICS.

Bertrand, G., *La théorie des marées el le calcul des variations*, ANNALES DE LA SOCIÉTÉ SCIENTIFIQUE DE BRUXELLES, vol. 41 (1921–22), p. 109.

Blaschke, *Geometrische Untersuchungen zur Variationsrechnung*, MATHEMATISCHE ZEITSCHRIFT, vol. 6 (1920), p. 281.

Carathéodory, *Über geschlossene Extremalen und periodische Variationsprobleme in der Ebene und im Raume*, ANNALI DI MATEMATICA, vol. 2 (1925), p. 297.

Courant, *Über ein Konvergenzerzeugendes Prinzip in der Variationsrechnung*, GÖTTINGER NACHRICHTEN, 1922, p. 144.

Courant, *Über direkte Methoden bei Variations- und Randwertproblemen*, JAHRESBERICHT DER VEREINIGUNG, vol. 34 (1925), p. 90.

Dunkel, *A determination of the curve minimizing the area inclosed by it and its evolute*, AMERICAN MATHEMATICAL MONTHLY, vol. 28 (1921), p. 15.

Dunkel, *A direct determination of the minimum area between a curve and its caustić*, ANNALS OF MATHEMATICS, (2), vol. 23 (1921–22), p. 135.

Dunkel, *The curve which with its caustic encloses the minimum area*, WASHINGTON UNIVERSITY STUDIES, vol. 8 (Scientific Series) (1921), p. 183.

Dunkel, *Integral inequalities with applications to the calculus of variations*, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), p. 326.

Funk and Berwald, *Flächeninhalt und Winkel in der Variationsrechnung*, LOTOS-PRAGUE, vol. 67 (1919–20), p. 45.

Radon, *Zur Behandlung geschlossener Extremalen in der Variationsrechnung*, ABHANDLUNGEN AUS DEM HAMBURGISCHEN SEMINAR, vol. 1 (1922), p. 195; see also ibid., vol. 4 (1925), p. 13.

Rider, *An intrinsic equation solution of a problem of Euler*, AMERICAN MATHEMATICAL MONTHLY, vol. 24 (1917), p. 420.

Valiron, *Sur un problème particulier de variations*, NOUVELLES ANNALES, (5), vol. 1 (1923), p. 196.

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