A NOTION OF UNIFORM INTEGRABILITY*

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The necessary and sufficient condition that a function $f(x)$ of the real variable x be integrable in the sense of Riemann on the interval (a,b) is that there correspond to an arbitrary small positive number ϵ a positive δ such that for any subdivision of (a,b) by points

$$x_0 = a \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1} \leqq x_n = b\ ,$$

subject to the condition $x_i - x_{i-1} < \delta$, the inequality

$$\sum_{i=1}^{n} (U_i - L_i)(x_i - x_{i-1}) \leqq \epsilon$$

is valid. In this, U_i and L_i represent, respectively, the upper and lower bounds of $f(x)$ on the subinterval (x_{i-1},x_i).

In the direct extension of this definition to a function which involves besides the variable of integration also other parameters, it may or may not be possible in any particular case to satisfy the conditions above by a constant δ independent of the parameters. In this connection the following concept may be of interest.

A function $f(x,\lambda)$ shall be defined to be integrable with respect to x on (a,b) uniformly in λ, provided that there corresponds to an arbitrary positive ϵ a positive constant δ independent of λ, such that

$$\sum_{i=1}^{n} \{U_i(\lambda) - L_i(\lambda)\}(x_i - x_{i-1}) \leqq \epsilon. \dagger \tag{1}$$

If $f(x,\lambda)$ is complex, it shall be said to be uniformly integrable if both its real and its imaginary parts are uniformly integrable.

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† The extension of this definition to the case when f involves a greater number of parameters, real or complex, is, of course, immediate.

As an application of this notion of uniform integrability we establish the following theorem.

THEOREM. *If the function $f(x,\lambda)$ of the real variable x and the complex parameter λ is, in the region*

$$0 \leqq a \leqq x \leqq b, \qquad R(\lambda) \leqq M \geqq 0\,,^*$$

defined, uniformly bounded, and integrable with respect to x, uniformly in λ, then the integral

$$I(f) = \int_a^b e^{\lambda x} f(x,\lambda)\,dx$$

approaches zero uniformly in λ as $|\lambda|$ becomes infinite.

It is sufficient for our purpose to give the proof for the real part only of the given function, $f^{(1)}(x,\lambda)$. Because of the uniform integrability of $f^{(1)}(x,\lambda)$, we may, when ϵ is assigned, determine δ so that (1) is satisfied. Then choosing on each subinterval (x_{i-1},x_i) a value $P_i(\lambda)$ subject to the conditions

$$L_i(\lambda) \leqq P_i(\lambda) \leqq U_i(\lambda)\,,$$

and defining the auxiliary stepfunction $\psi(x,\lambda)$ as the function taking the value $P_i(\lambda)$ on (x_{i-1},x_i) we have

$$I(f^{(1)}) = I(\psi) + I(f^{(1)} - \psi)\,.$$

Now on the one hand

$$|I(\psi)| = \left| \sum_{i=1}^{n} P_i(\lambda) \cdot \frac{e^{\lambda x_i} - e^{\lambda x_{i-1}}}{\lambda} \right|,$$

whence, since $|P_i(\lambda)| \leqq k$ because $f^{(1)}$ is uniformly bounded,

$$|I(\psi)| \leqq \frac{2kne^{Mb}}{|\lambda|}\,.$$

On the other hand since $f^{(1)}$ is uniformly integrable

$$|I(f^{(1)} - \psi)| \leqq e^{Mb} \int_a^b |f^{(1)}(x,\lambda) - \psi(x,\lambda)|\,dx$$

$$\leqq e^{Mb} \sum_{i=1}^{n} \{ U_i(\lambda) - L_i(\lambda) \} (x_i - x_{i-1}) \leqq e^{Mb}\epsilon\,.$$

* If these restrictions on x and λ are omitted, the reasoning employed leads to the result that $I(f) = e^{\lambda a}\epsilon(\lambda) + e^{\lambda b}\epsilon(\lambda)$, where each $\epsilon(\lambda)$ denotes some function which approaches zero uniformly as $\lambda \to \infty$. The symbol $R(\lambda)$ designates the real part of λ.

Hence

$$|I(f^{(1)})| \leqq e^{Mb}\left[\frac{2nk}{|\lambda|}+\epsilon\right],$$

and since n is fixed when ϵ is given, the theorem is proved.

The situation changes materially when integrability is considered in the sense of Lebesgue. By definition, then, the integral of f is equal to the limit as $\delta \to 0$ of the series

$$\sum_{k=-\infty}^{\infty} y_k m E_x(y_{k-1} < f(x,\lambda) \leqq y_k), \tag{2}$$

where $\cdots y_{-2} < y_{-1} < y_0 < y_1 < y_2 \cdots$ denotes an arbitrary subdivision of the range of functional values subject to the condition $y_k - y_{k-1} < \delta$. It would seem natural to define the integrability of $f(x,\lambda)$ as uniform, if for a given δ the approximation to the integral given by (2) is uniformly good. This fact, however, is already contained implicitly in the definition of the integrability.*

The theorem proved above is, however, not true if $f(x,\lambda)$ is merely integrable in the sense of Lebesgue. A further condition on the function must be imposed. As an example of such a condition we mention the following, that $f(x,\lambda)$ may be uniformly approximated to by a sequence of functions $f_i(x,\lambda)$, each of which is integrable with respect to x (in the sense of Riemann) uniformly in λ, the approximation being uniform in the sense that there corresponds to any given $\epsilon > 0$, a constant i_0 independent of λ such that

$$\int_a^b |f(x,\lambda) - f_i(x,\lambda)|\,dx < \epsilon \quad \text{for} \quad i \geqq i_0 .$$

In the important special case when f depends only on x this condition is always satisfied.

It should be noticed that the condition above does not suppose that the function $f(x,\lambda)$ is bounded.

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* Cf. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig-Berlin, 1918, pp. 450–453.