THE ALTERNATION OF NODES OF LINEARLY INDEPENDENT SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS*

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We shall consider solutions of the difference equation

$$(1) \qquad u(n+2)=A(n)\,u(n+1)-B(n)\,u(n), \qquad B(n)>0,$$

where $A(n)$ and $B(n)$ are finite and single-valued functions of the integer n. If the points obtained by plotting a solution $u(n)$ are joined by segments of a straight line, this broken line gives a representation of a single-valued and continuous function $f(x)$ such that $f(n)=u(n)$. The zeros of $f(x)$ are called the nodes of $u(n)$.

Proofs have already been given of the following theorem.

THEOREM. *The nodes of two linearly independent solutions of* (1) *separate one another.*†

The proof which is to be given here seems simpler and more obvious than either of these two proofs. Two known and easily verified facts will be used. If $u_1(n)$ and $u_2(n)$ are any two solutions of (1) and if we set

$$\Delta(n)=\begin{vmatrix} u_1(n) & u_2(n) \\ u_1(n+1) & u_2(n+1) \end{vmatrix},$$

then

$$\Delta(n+1)=B(n)\Delta(n)\ .$$

As a first result of the condition imposed upon $B(n)$ in (1) we have the fact that if $\Delta(n)$ is not zero for one value of n then it is never zero and its sign remains unchanged.

A necessary and sufficient condition that the two solutions $u_1(n)$ and $u_2(n)$ are linearly independent is that $\Delta(n)$ is not zero for one value of n.

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† Porter, ANNALS OF MATHEMATICS, (2), vol. 3, (1901–02), p. 65. Moulton, E. J., ibid. ,(2), vol. 13 (1911–12), p. 137.

We have

$$f_i(x)=(x-n)[u_i(n+1)-u_i(n)]+u_i(n), \quad n\leqq x<n+1, \quad i=1,2 .$$

An easy calculation shows that

$$(2) \qquad W(x)=\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}=\Delta(n) , \qquad n<x<n+1 .$$

If we set

$$(3) \qquad f_i'(n)=u_i(n+1)-u_i(n) ,$$

then $W(x)$ is defined for all values of x, and it is never zero and it has always the same sign, if, as we shall now suppose, $u_1(n)$ and $u_2(n)$ are linearly independent.

We shall now have to notice a second result of the condition placed upon $B(n)$. If $u(n+1)=0$, $u(n+2)=-B(n)u(n)$, and hence $f'(x)$, which may be discontinuous at $n+1$, has the same sign a little before, at, and a little after $n+1$. The case in which $u(n)$ is zero for two successive values of n cannot occur here for two linearly independent solutions.

Suppose that $f_1(x)$ vanishes at x_1 and at x_2 but at no point between. Then

$$(4) \qquad f_1'(x_1)f_1'(x_2)<0 ,$$

even if one or both of x_1 and x_2 are integers as a result of the remark above. Then from the facts stated above

$$(5) \qquad W(x_1)\,W(x_2)=f_1'(x_1)f_2(x_1)f_1'(x_2)f_2(x_2)>0 .$$

From (4) and (5) it follows that

$$f_2(x_1)f_2(x_2)<0 ,$$

and this requires that $f_2(x)$, which is continuous, shall vanish at least once between x_1 and x_2. It cannot vanish more than once, for if it did then the same form of proof would show that $f_1(x)$ vanishes between x_1 and x_2 contrary to the hypothesis. This concludes the proof.

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