COVERING THEOREMS*

BY R. L. MOORE

A set of segments will be said to cover a point set F *in the Vitali sense* if, for every point P which belongs to F and every positive number e, there exists a segment of G which contains P and is of length less than e. J. Splawa-Neyman has shown† that if, in space of one dimension, F is a closed and bounded point set of measure zero and G is a set of segments which covers F in the Vitali sense, then, for every positive number e, the set G contains a subset G_e such that G_e covers F and such that the sum of the lengths of the segments of the set G_e is less than e. He cites the question, raised by Sierpinski, whether this theorem remains true after the removal of the condition that F be closed. I have recently‡ answered this question in the negative. Splawa-Neyman shows, by an example, that his theorem does not hold true for two dimensions, but makes the following statement, without proof:

"Remarquons que notre théorème subsiste pour les espaces à n dimensions, s'il existe pour tout point p de F une sphére appartenant à F de rayon aussi petit que l'on veut et *dont le centre est en p.*"

In the present paper I will show that the theorem thus stated, without proof, by Splawa-Neyman remains true on the removal of the condition that F be closed. I will also show that the condition that each point of F be the center of spheres of

G of arbitrarily small radius may be replaced by a weaker condition. These results will be established with the aid of certain theorems* which may, perhaps, be of some interest in themselves. I will first consider these theorems.

THEOREM 1. *If $0 < k < 1$, $e < 0$, and G is a set of circular regions†* *all of radius greater than e and, for every region g of the set G, h_g is a circular region, concentric with g and with a radius equal to k times the radius of g and H is the set of all such regions h_g for all regions g of the set G and the sum of all the regions of the set G is a bounded point set, then the set G contains a finite subset $\overline{G}$ such that every region of the set H lies wholly in some region of the set $\overline{G}$.*

PROOF. Let K denote the point set composed of the centers of the regions of the set G. Let T denote the set of all circular regions t, of radius equal to $(1-k)e/4$, such that t is concentric with some region of the set H. It is easy to see that the closed point set $K+K'$, composed of K plus its derived set, is covered by T. It follows, by the Borel-Lebesgue theorem, that T contains a finite subset $\overline{T}$ which covers $K+K'$. For each region t of the set $\overline{T}$ select just one region of G whose center is in t and whose radius is greater than $r_t - (1-k)e/2$, where r_t is the least upper bound of the radii of all those regions of G whose centers are in t. Let $\overline{G}$ denote the set of all regions so selected. It is easy to see that every region of the set H lies wholly in some region of set $\overline{G}$.

THEOREM 2. *If k is a positive number less than unity and G is a set of circles and there exists no positive number such that infinitely many circles of the set G have radii greater than that number and, for each circle g of the set G, h_g is a circle concentric with g and with a radius equal to k times that of g, and H is the totality of all such circles h_g for all circles g of the set G, then there exists a subset $\overline{G}$ of the set of circles G such that (a) each circle of the set H is within some circle of the set $\overline{G}$, but (b) no circle $\overline{g}$ of*

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* These theorems will be stated in the terminology of space of two dimensions. It is easy to see however that this restriction is not necessary.

† By a *circular region* is meant the interior of a circle.

the set $\overline{G}$ completely encloses any circle h, of the set H, such that $h = h_g$ *for some circle g, other than $\overline{g}$, of the set $\overline{G}$.*

PROOF. If for any circle g of the set G there exists another circle of G which is concentric with it let k_g denote the largest circle of G which is concentric with g and let K denote the set of all such circles k_g for all circles g of the set G. Let $\overline{H}$ denote the set of all circles h of the set H such that $h = h_g$ for some circle g which belongs to K. Clearly every circle of H either is identical with some circle of $\overline{H}$ or is within, and concentric with, some circle of $\overline{H}$. No two circles of $\overline{H}$ are concentric with each other. It follows from the hypothesis that there exists a circle g_1 which belongs to the set K and which has as large a radius as any other circle of the set K. Let G_1 denote the set of circles consisting of g_1 together with every circle of K which is not concentric with any circle of $\overline{H}$ which lies within g_1. No circle of G_1, except g_1, encloses h_g, and g_1 does not enclose any circle of the set $\overline{H}$ (except h_{g_1}) which is concentric with a circle of the set G_1. Furthermore every circle of the set H is within some circle of the set G_1. If G_1 contains any circle other than g_1 there exists, in the set G_1, a circle g_2, distinct from g_1 and with a radius as great as that of any other circle in G_1 except g_1. Let G_2 denote the set of circles consisting of g_2 together with every circle of the set G_1 which is not concentric with any circle of the set $\overline{H}$ which lies within g_2. Clearly g_1 belongs to G_2. Continue this process thus obtaining a finite or countably infinite sequence of circles g_1, g_2, g_3, $\cdots$ and a corresponding sequence of sets of circles G_1, G_2, G_3, $\cdots$ such that, if $n > 1$, G_n consists of g_n together with every circle of the set G_{n-1} which is not concentric with any circle of the set $\overline{H}$ which lies within g_n and if G_n contains any circle other than g_1, g_2, g_3, $\cdots$ g, g_n then g_{n+1} is a circle belonging to G_n and distinct from the circles g_1, g_2, g_3, $\cdots$, g_n and with a radius as great as that of any other circle in G_n except g_1, g_2, g_3, $\cdots$, g_n, but if G_n contains no circle other than g_1, g_2, g_3, $\cdots$, g_n (i.e. if every circle belonging to G_{n-1}, except g_1, g_2, g_3, $\cdots$, g_n, is concentric with some circle of the set $\overline{H}$ which lies within g_n)

then the sequences g_1, g_2, g_3, $\cdot$ $\cdot$ $\cdot$ and G_1, G_2, G_3, $\cdot$ $\cdot$ $\cdot$ are finite and g_n and G_n are their last terms. The finite or countably infinite set of circles g_1, g_2, g_3, $\cdot$ $\cdot$ $\cdot$ satisfies all the requirements indicated for the set $\overline{G}$.

THEOREM 3. *If k_1 and k_2 are positive numbers $(1 > k_2 > k_1)$ and D is a bounded domain and G is a set of circular regions lying in D and, for every region of the set G, h_g is a circular region concentric with g and with a radius k_1 times that of g and H is the set of all regions h_g for all regions g of the set G, then there exists a subset $\overline{G}$ of the set G such that every region of the set H is contained in some region of the set $\overline{G}$ and such that the sum of the areas of all the regions of the set $\overline{G}$ is less than or equal to $4A/(1-k_2)^2$, where A is the area of the domain D.*

PROOF. For each circle g of the set G construct a circular region t_g concentric with g and with a radius equal to k_2 times that of g. Let T denote the set of all the regions t_g for all regions g of the set G. Let H_1 denote the set of all those regions of the set H whose radii are greater than or equal to 1. For each positive integer n, greater than 1, let H_n denote the set of all those regions of the set H whose radii are greater than or equal to $1/n$ but less than $1/(n-1)$. By Theorem 1, for each n, there exists a finite subset T_n of the set of regions T such that every region of the set H_n is a subset of some region of the set T_n. Let $\overline{T}$ denote the set of regions composed of all the regions of all the sets T_1, T_2, T_3, $\cdot$ $\cdot$ $\cdot$. There exists no positive number such that infinitely many regions of the set $\overline{T}$ have radii greater than that number. It follows, by Theorem 2, that there exists a subset $\overline{G}$ of the set of regions G such that (a) every region of the set $\overline{T}$ is a subset of some region of the set $\overline{G}$, (b) no region $\overline{g}$ of the set $\overline{G}$ contains any region t of the set $\overline{T}$ such that $t = t_g$ for some region g, other than $\overline{g}$, of the set $\overline{G}$. For each region g of the set $\overline{G}$ let q_g denote a circular region concentric with g and having a radius equal to the radius of g multiplied by $(1-k_2)/2$. Let Q denote the set of all such regions q_g for all regions g of the set $\overline{G}$. It is easy to see that no two regions of the set Q have a point in common. Hence the

sum of the areas of all the regions of this set is less than or equal to A. It follows that the sum of the areas of all the regions of the set $\overline{G}$ is less than or equal to $4A/(1-k_2)^2$. But every region of the set H is a subset of some region of the set $\overline{T}$ and therefore of some region of the set $\overline{G}$. The truth of Theorem 3 is therefore established.

THEOREM 4. *If M is a bounded point set of measure zero and k is a positive number less than* 1 *and G is a set of circular regions and for every point P of M and every positive number e there exists, in the set G, a region g of radius less than e and such that the distance from P to the center of g is less than k times the radius of g, then, for every positive number e, there exists a subset $\overline{G}$ of G such that $\overline{G}$ covers M and such that the sum of the areas of the regions of the set $\overline{G}$ is less than e.*

PROOF. For each region g of the set G let h_g denote a circular region concentric with g and with a radius k times that of g. Let H denote the set of all such regions h_g. Let $\overline{k}$ denote some definite number between 1 and k. Suppose e is a positive number. Since M is of measure zero there exists a set S of circular regions covering M and such that the sum of the areas of the regions of S is less than $e(1-\overline{k})^2/4$. For each point P of M there exists a circular region g belonging to the set G and lying in some region of S and such that P lies in the region h_g. It follows by Theorem 3 that there exists a subset $\overline{G}$ of G such that each region of H is a subset of some region of $\overline{G}$, and the sum of the areas of the regions of the set $\overline{G}$ is less than or equal to $4B/(1-\overline{k})^2$, where B is the sum of the areas of the regions of the set S. Since B is less than $e(1-k)^2/4$ it follows that the sum of the areas of the regions of the set $\overline{G}$ is less than e.

DEFINITIONS. By a *simple chain of segments* in space of one dimension is meant a set C of segments such that (a) the point set obtained by adding together the points of all the segments of C is connected, (b) there exists no point which is common to more than two segments of C, and (c) no segment of C is a subset of any other segment of C.

By a *simple chain of intervals* in space of one dimension is meant a set C of intervals such that (a) the point set obtained by adding together the points of all the intervals of C is connected, (b) there exists no point which is common to more than two intervals of the set C, (c) no interval of C is a part of any other interval of C, and (d) no endpoint of an interval of C is a limit point of a set of endpoints of intervals of C.

THEOREM 5. *If, in space of one dimension, G is a set of $\begin{Bmatrix} segments \\ intervals \end{Bmatrix}$ and there exists no positive number such that infinitely many $\begin{Bmatrix} segments \\ intervals \end{Bmatrix}$ of the set G are of length greater than that number then G contains a subset $\bar{G}$ such that (a) every point that belongs to $\begin{Bmatrix} a\ segment \\ an\ interval \end{Bmatrix}$ of G belongs also to some $\begin{Bmatrix} segment \\ interval \end{Bmatrix}$ of $\bar{G}$, (b) G consists of a set of nonoverlapping simple chains of $\begin{Bmatrix} segments. \\ intervals. \end{Bmatrix}$*

The truth of Theorem 5, for the case where G is a set of segments, is established in the course of the argument given to prove Theorem 1 of my PROCEEDINGS paper referred to above. For the case where G is a set of intervals the same argument will apply if the word "segment" is replaced, at times by the word "interval" and at times by the phrase "other interval."

THEOREM 6. *If, in space of one dimension, K is a bounded point set of measure zero and G is a set of intervals such that, for each positive number e, each point of K belongs to at least one interval of G of length less than e and there exists no positive number such that infinitely many intervals of the set G are of length greater than that number, then, for every positive number e, G contains a finite, or countably infinite, subset G_e such that G_e covers K and such that the sum of the lengths of the intervals of G_e is less than e and such that, furthermore, G_e consists of a countable number of nonoverlapping simple chains of intervals.*

With the help of Theorem 5, Theorem 6 may be proved by an argument similar to that used to prove Theorem 1 of the above mentioned paper.

Splawa-Neyman indicates that his theorem becomes true for a point set F of measure greater than zero if the condition that the sum of the lengths of the segments of the set G_e be less than e is replaced by the condition that it be less than $m(F)+e$, where $m(F)$ is the measure of F. Theorems 4 and 6 of the present paper may be generalized in an analogous manner. The truth of this statement for the case of Theorem 6 may be seen with the help of the fact that, under the hypothesis of this theorem (modified by the omission of the requirement that K be of measure zero), the set G contains as a subset a set S of *nonoverlapping* intervals such that (a) every interval of S lies in a certain predetermined domain whose measure exceeds the measure of F by less than $e/2$, (b) S covers the whole of F except for a point set of measure zero. The existence of such a set S follows from Carathéodory's generalization* of Vitali's theorem. A largely similar argument holds good for the case of Theorem 4.

That Splawa-Neyman's theorem does not remain true if the words "à son intérieur" are omitted may be seen as follows. Using the notation of the example given on pages 464 and 465 of my PROCEEDINGS paper, for each positive integer n let K_n denote the point set consisting of the left ends of all the segments of the set G_n. For each point P of K_n let I_P denote the smallest segment of G_n which has P as its left end and let H_P denote a set of intervals having P as their common left end, their right ends forming an infinite sequence of points A_{P1}, A_{P2}, A_{P3}, $\cdots$ such that, for every positive integer j, the length of the interval whose end points are P and A_{Pj} is equal to the length of I_P divided by $2j$. For each n let Q_n denote the set of intervals composed of all the intervals of all the sets H_P for all points P of K_n. Let Q denote the set of intervals composed of all the intervals of all the sets Q_1, Q_2, Q_3, $\cdots$, together with a set of intervals having B as their common left end, their right ends forming an infinite sequence of points B_1, B_2, B_3, $\cdots$, all lying to the right of B and such that the

* Cf. C. Carathéodory, *Vorlesungen über reelle Funktionen*, Teubner, 1918, p. 304.

distance from B_j to B is equal to $1/j$. Let W denote the set of intervals composed of all the intervals of Q and all intervals t such that t is equal to some segment of G plus its endpoints. Let F denote the point set $B+K+K_1+K_2+K_3+ \cdot \cdot \cdot$. It can be seen that F is a closed point set of measure zero and that it is covered in the Vitali sense by the set of intervals W. But W contains no subset $\overline{W}$ such that $\overline{W}$ covers F and such that the sum of the lengths of the intervals of $\overline{W}$ exists (as a finite number).

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THE CONVERGENCE OF A GENERAL MEAN OF MEASUREMENTS TO THE TRUE VALUE*

BY E. L. DODD

THEOREM. *For positive measurements x_i suppose that the frequency function $\phi(x)$ has a positive lower bound b in some interval $(a-k, a+k)$ about the true value a, and that for each positive r*

$$(1) \qquad m_r = \int_0^\infty x^r \phi(x) dx , \qquad (finite); \qquad m_0 = 1.$$

Then there exists a continuous increasing function $f(x)$, with inverse f^{-1}, such that if $\epsilon > 0, \eta > 0$, and

$$(2) \qquad M = f^{-1}\{ Arithmetic \ mean \ of \ f(x_i) \}, \quad i = 1, 2, \cdots, n,$$

then, when $n >$ some n', there is a probability $> 1 - \eta$ that

$$(3) \qquad |M-a| < \epsilon.$$

PROOF. For (3) the special condition† is

$$(4) \qquad f(a) = \int_0^\infty f(x)\phi(x) \ dx .$$

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† Dodd, *Functions of measurements under general laws of error*, SKANDI-NAVISK AKTUARIETIDSKRIFT, vol. 5 (1922), pp. 133–158. Apply theorem, p. 135, taking $\Sigma f(x_i) < nf(a+\epsilon) = n(1+\delta)f(a) = z$; then $\Sigma f(x_i) < nf(a-\epsilon)$.