ON IRREDUNDANT SETS OF POSTULATES*

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In his paper *On irredundant sets of postulates*,† Mr. Alonzo Church gives a *mechanical method*‡ by which any set of postulates can be made irredundant. This method in the general case is as follows. Given a set of postulates $A_1, A_2, \cdots, A_n$. Form the set of postulates $B_1, B_2, \cdots, B_n$, where $B_1=A_1$ and for each $i (i= 2, 3, \cdots, n)$, B_i denotes the proposition *if* $A_1, A_2, \cdots, A_{i-1}$, *then* A_i.

Obviously the negatives of any two postulates of the set [B] are contradictory. Hence to show that the set [B] is irredundant, we need merely show the postulates independent by showing for each $i (i=1, 2, \cdots, n)$, an example in which B_i is false. This requires the existence of examples exhibiting these characteristics in terms of the set [A]: $A_1, A_2, \cdots, A_{i-1}$ true, A_i false, for each i.

Even if the postulates of set [A] are not independent, the postulates of set [B] are independent (and irredundant), except when a relation exists of this form:

$$\text{(I)} \quad \text{If } A_{n_1}, A_{n_2}, \cdots, A_{n_{k-1}}, \text{ then } A_{n_k}, \qquad \text{for } 1 \leqq n_1 < n_2 < \cdots < n_{k-1} < n_k \leqq n,$$

in which case the postulates of set [B] are not independent.

We have here a new method of obtaining independence among postulates. *Given any set of n postulates* [A] *which can be arranged in a sequence such that no relation of form* (I) *exists. The set* [A] *can be replaced, without losing any implications, by*

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* Presented to the Society, October 31, 1925.

† TRANSACTIONS OF THIS SOCIETY, vol. 27 (1925), p. 318. A set of postulates is *irredundant* if the postulates are independent and the negatives of every two are contradictory.

‡ Loc. cit., p. 321. Church confines his remarks to the case where the postulates are independent.

the set of n postulates [B], obtained as described above, which are irredundant and therefore independent. If anyone desires to use this method, he has the author's permission to do so.

We shall give a few examples in which this method may have been used. In the examples, we shall use the following abbreviations: N is the set of all numbers. An S-set is a set of numbers (not null) such that every number in the set has a successor in the set. A T-set is a set of numbers (not null) such that every number in the set is successor of some number in the set. An ST-set is a set which is both an S-set and a T-set.

EXAMPLE 1. Church* gives an example of an irredundant set of postulates for a system of a finite number of elements arranged in cyclic order. Using the above notation, it is easily seen that these postulates are the following:

B_1. An S-set exists.

B_2. If an S-set exists, N is the only S-set.

These evidently may be derived by the mechanical method from the following postulates:

A_1. An S-set exists.

A_2. N is the only S-set.

Since A_2 implies A_1, no implications are lost if we replace the set of postulates [A], or the set [B], by the single postulate A_2.

EXAMPLE 2. Church† gives an example of an irredundant (and categorical) set of postulates for the system of positive and negative integers. Using the above notation, it is easily seen that these postulates are the following:

C_1. An S-set exists.

C_2. If an S-set exists,‡ some proper part of N is an S-set.

C_3. If an S-set exists, an ST-set exists.

C_4. If an ST-set exists, N is the only ST-set.

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* Loc. cit., p. 321.

† Loc. cit., p. 323.

‡ We can evidently replace the assumption *if N is an S-set*, by the assumption *if an S-set exists*, without adding or losing any implications.

Since the hypothesis of C_4 carries with it the hypothesis *if an S-set exists*, it is evident that the postulates C_1, C_3, C_4 may be derived by the mechanical method from the following postulates:

A_1. An S-set exists.

A_2. An ST-set exists.

A_3. N is the only ST-set.

Since A_3 implies A_2, and A_2 implies A_1, no implications are lost, if we replace A_1, A_2, A_3 or C_1, C_3, C_4 by the single postulate A_3. That is, the set of postulates A_3, C_2 yields the same implications as the set [C]. Moreover, this set is irredundant, since the logical relation exists that if N is the only S-set, then N is the only ST-set.*

But if we desire a categorical set of postulates for the system of positive and negative integers, that is independent but not irredundant, (which in this case implies that the set is completely independent), such a set is the set consisting of A_3 and the conclusion of C_2, that is:

D_1. N is the only ST-set.

D_2. Some proper part of N is an S-set.

As Church† has pointed out, the question arises to what extent it is possible to obtain irredundant sets of postulates which are not open to the objection that they are formed by the mechanical method. Both of the above examples are open to this objection, and to the further objection that they are formed by the mechanical method from a set of postulates which are not independent.

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* Church, loc. cit., p. 323.

† Loc. cit., p. 321.