PROPERTIES OF UNRESTRICTED REAL FUNCTIONS*

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The functions that ordinarily interest mathematicians are of specialized character—continuous, differentiable, analytic, of limited variation, etc. Historically, of course, the race starts with simple, concrete things and only gradually moves on to abstract conceptions. In the case of functions, it was not till the nineteenth century that serious attention was accorded functions affected with a generous degree of discontinuity, and it was not till the middle of the century that there emerged Dirichlet's conception of an unrestricted (real) function. According to this conception, $g(x)$ is a real function of the real variable x if to every real number x there corresponds a real number $g(x)$. This conception, natural and simple though it is, conflicted with the traditional notion —which, indeed, is the same as the one widely held by those unconversant with modern developments of the theory of functions of a real variable—that required from every function some sort of analytic expressibility. Thus, $g(x)$ is a function, according to Dirichlet, if $g(0)=0$ and $g(x)=1$ for $x\neq 0$. It so happens that this particular $g(x)$ is analytically expressible as

$$\lim_{n\to\infty} \left(x^{\frac{1}{2n+1}}\right)^2$$

for example, where n is a positive integer. But the decision that $g(x)$ is a function rests, for Dirichlet, solely on the ground of the correspondence of a real number $g(x)$ to every real number x; whereas, according to the older conception,

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it was necessary to demonstrate the expressibility of $g(x)$ by means of sums, products, sines, logarithms, limits, etc., before admitting it into the domain of functions.

The functions we shall deal with are single-valued—otherwise they are unrestricted. The assumption of one-valuedness may be regarded as entering the definition of function, if we require that to every x there shall correspond just one number $g(x)$; besides, the properties we shall be concerned with hold essentially even when this assumption is dropped.

Now suppose one were to select a particular function $g(x)$ out of the class of unrestricted one-valued, real functions. If we are told nothing concerning his choice, what can we say about $g(x)$? Anything but trivialities? Compare the question with the following one: Suppose one were to select a real number without telling us anything concerning it. What can we say about it? In this case, I know of nothing insufficiently evident to be worth while asserting.

Not so, however, with the question concerning the function $g(x)$. In other words, we *are* able to state properties—by no means trivial—that hold for *all real functions*. This may seem unexpected. Even Hobson, in his latest edition of *The Theory of Functions of a Real Variable* (1921) writes: "No elaborate theory is required for functions which retain their complete generality, . . . since few deductions of importance can be made from that definition which will be valid for all functions."

A property of all real functions $g(x)$ can, of course, be a consequence of nothing else than that $g(x)$ is a real function. What may, however, not be seen at once is that the condition that $g(x)$ is a real function has implications that are far from obvious and yet of a simplicity that attracts interest. There is thus added to our conception of an unconditioned function a richness of detail that one would at first hardly suspect.

It so happens that the subject matter of this paper lies near the foundations of mathematical analysis, so that only a slight knowledge of mathematics is necessary for understanding the content of the theorems—if not of the proofs, which we mostly omit anyway. To make the paper intelligible to a wider circle of readers, we shall assume no further technical knowledge on the part of the reader than that possessed by one who knows the calculus.

We first explain the notions *least upper bound*, *greatest lower bound*, and *saltus*. The *least upper bound* of a real, one-valued function $g(x)$ in an interval (a, b) is, as the very name signifies, the smallest number not exceeded by the functional values of $g(x)$ in (a, b); we denote it by $u(g, a, b)$. Thus suppose $g(x)$, defined for $0 \leqq x \leqq 1$, is 0 for $x=0$ and also for irrational values of x; and $g(x)=x-1/q$ if $x=p/q$, a rational fraction in its lowest terms. Then $u(g, 0, 1)=1$, because 1 is the smallest number not exceeded by the values of $g(x)$. Similarly, the greatest lower bound of $g(x)$ in an interval (a, b), denoted by $l(g, a, b)$ is the largest number exceeding no value of $g(x)$ for $a \leqq x \leqq b$. The *saltus* of $g(x)$ in (a, b), denoted by $s(g, a, b)$, is the span of variation of the functional values of $g(x)$ in the interval, and is defined by the equation

$$s(g, a, b)=u(g, a, b)-l(g, a, b).$$

We have defined the saltus in an interval (a, b); we now define the saltus at a point x—the passage from the former to the latter being made like that from average speed in an interval of time to instantaneous speed. We enclose the point x in intervals (α, β) of length $l_{\alpha\beta}$ approaching 0, and define the saltus of $g(x)$ at x, which we denote by $s(g, x)$, as

$$\lim_{l_{\alpha\beta}\to 0} s(g, \alpha, \beta),$$

where (α, β) is understood to be an interval containing x as interior point. It may be seen after slight reflection that this limit always exists. In the case of the particular function $g(x)$ defined above, $s(g, x)=x$, $0 \leqq x \leqq 1$.

With every function $g(x)$—no matter how discontinuous it may be—we can thus associate a species of "derived" function, namely, the saltus function $s(g, x)$. The successive saltus functions derived from a given primitive $g(x)$ we denote by

$$s'(g, x)=s(g, x), \quad s''(g, x)=s(s', x), \quad s'''(g, x)=s(s'', x),$$

and so on.

We are now ready to tell something about the selected function $g(x)$ without inquiring as to its special character; namely, that $s'''(g, x)\equiv s''(g, x)$.* In other words, in the succession of saltus functions $s'(g, x)$, $s''(g, x)$, $s'''(g, x)$, $\cdots$ all from the second on are equal. Here is a concrete example. Let $g(x)$, defined in the interval $(0, 1)$, be equal to x for irrational x; $g(x)=0$ for $x=0$, $x=1$, and for rational fractions in which the denominator is an integral power of 2; $g(x)=x+1/q$ for every rational fraction $x=p/q$, supposed to be in its lowest terms, with q not an integral power of 2. It may be seen that $l(g, x)=0$ for every x, and that $u(g, x)=x$ everywhere except that $u(g,p/q)=(p+1)/q$ for fractions p/q of the described type. Hence $s'(g, p/q)=(p+1)/q$, and $s'(g, x)=x$ for every x not of the form p/q. Therefore $l(s', x)$, i. e., the lower bound function of the first saltus function, equals x at every point, and $u(s', x)=x$ except that $u(s',p/q)=p+1/q$, so that $s''(g, p/q)=1/q$, and $s''(g, x)=0$ for $x\neq p/q$. Hence $s'''(g, p/q)=1/q$ and $s'''(g, x)=0$ for $x\neq p/q$, so that $s'''(g, x)\equiv s''(g, x)$.

To be able to tell more about unconditioned functions, we shall define other types of saltus. These new types are obtained by agreeing to regard certain point sets as negligible The first new type thus obtained, called the *f-saltus*, comes by considering *finite* sets as negligible. The f-saltus of $g(x)$ in the interval (a, b), denoted by $s_f(g, a, b)$, is defined as the number k satisfying the following two conditions: (a) by neglecting the values of $g(x)$ in a suitably chosen finite set of points, we can make the resulting ordinary saltus $s(g, a, b)$ less than k plus as small a number as we please; (b) no matter what finite set we choose to neglect, we cannot make the resulting ordinary saltus less than k. Thus suppose $g(x)$,

* Sierpiński, BULLETIN DE L'ACADÉMIE DES SCIENCES DE CRACOVIE (1910), pp. 633-634.

defined in the interval (0, 1), equals 0 everywhere except at the points $x=1/2^n$, $n=1, 2, \cdots$, where $g(x)=x$. The ordinary saltus in the interval is 1/2. But if we exclude the point $x=1/2$, the resulting saltus is 1/4; if we exclude the points 1/2 and 1/4, the resulting saltus is 1/8; and so on. Hence $s_f(g, 0, 1)=0$, while $s(g, 0, 1)=1/2$.

As in the case of the ordinary saltus, we pass from the idea of the f-saltus in an interval to that of the f-saltus at a point x by means of a sequence of intervals enclosing x and having 0 as limiting length. With every function $g(x)$, there thus coëxists, apart from $s(g, x)$, this new f-saltus function, which we denote by $s_f(g, x)$.

We obtain another type of saltus, the *d-saltus*, by agreeing that *denumerable* sets may be neglected; i. e., infinite sets containing as many individuals as there are positive integers; or in other words, sets whose elements may be completely mated in one-to-one manner with all the positive integers. For example, the set of rational numbers between 0 and 1 is denumerable; for we can arrange all the rational numbers in the intervals (0, 1) as follows:

0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, $\cdots$,
and then mate them successively with the positive integers, the 0 with the 1, the 1 with 2, the 1/2 with the 3, and so on, every rational number between 0 and 1 being thus mated to just one positive integer, and every positive integer to just one rational number. Let $g(x)=x$ for x rational, and $g(x)=0$ for x irrational. Then $s(g, 0, 1)=1$ and $s_f(g, 0, 1)=1$. But $s_d(g, 0, 1)=0$, i. e., the d-saltus of $g(x)$ in the interval (0, 1) is 0, since we may neglect the functional values at the denumerable set of points where x is rational.

Another type of saltus one naturally thinks of is obtained by regarding sets of *zero measure* (Lebesgue) as negligible. A set is said to be of zero measure if it can be enclosed in a denumerable infinity of intervals in such a way that the sum of the lengths is arbitrarily small. For example, suppose that a point set S on a straight line is denumerable, so that the totality of points in S may be mated in one-to-one manner with the positive integers. In this one-to-one correspondence, let P_n denote the element of S that is mated with the positive integer n. Enclose P_n $(n=1, 2, \cdots)$ in an interval of length $\delta/2^n$, where δ is a positive number. The totality of points of S, i. e., the points of the sequence $P_1, P_2, \cdots$, may therefore be covered by intervals the sum of whose lengths is

$$\frac{\delta}{2}+\frac{\delta}{2^2}+\cdots=\delta .$$

Since δ can be made arbitrarily small, the "length" of the set S is zero; that is to say, S is of zero measure (Lebesgue) according to our definition.

We may remark that the reasoning above shows that it is impossible to mate *all* the points of the continuum, or for that matter, all the points in an interval (a, b), with the positive integers in one-to-one manner. For if the points in (a, b) were denumerable, we could cover them completely by means of intervals whose sum of lengths is arbitrarily small. This would involve contradiction in our notions of length. We have here, then, a proof of the theorem of Cantor that it is impossible to arrange all the points of the continuum in an ordered sequence like that of the positive integers.

If we may neglect sets of zero measure, we are led to the z-saltus. The set of rational points between 0 and 1, constituting, as we have seen, a denumerable set, is of measure zero. Suppose $g(x)=0$ for irrational points and 1 for rational points of the interval $(0, 1)$. Then $s(g, 0, 1)=s_f(g, 0, 1)=1$ and $s_d(g, 0, 1)=s_z(g, 0, 1)=0$; here $s_z(g, 0, 1)$ denotes the z-saltus of $g(x)$ in the interval $(0, 1)$. While every denumerable set is of zero measure, the variety of sets of zero measure is so great that the denumerable sets may be said to constitute but a vanishing portion of the totality of sets of zero measure. Suppose Z is a non-denumerable set of points of zero measure lying in the interval (a, b); let $g(x)=0$ if x belongs to Z, and $g(x)=1$, if x belongs to the interval (a, b) but not to Z. Then $s_d(g, a, b)=1$ while $s_z(g, a, b)=0$.

Besides the ordinary saltus function, we thus have the f-saltus function, the d-saltus function and the z-saltus function, and for each of these new types we have a property of unconditioned functions similar to the one for the ordinary saltus. In the case of the d-saltus and the z-saltus, we have

$$s_d''' (g, x) \equiv s_d^{\mathrm{IV}} (g, x) \;,$$

and

$$s_z''' (g, x) \equiv s_z^{\mathrm{IV}} (g, x) \;;*$$

while examples of functions $g(x)$ may be constructed for which $s_d'' (g, x) \not\equiv s_d''' (g, x)$ and $s_z'' (g, x) \not\equiv s_z''' (g, x)$. The corresponding result for the f-saltus is more complicated; we omit its statement here.†

It may be remarked that $g(x)$ may be many-valued without significantly disturbing the general properties already mentioned; likewise, infinite values of $g(x)$ may be admitted without invalidating the results, provided familiar agreements are made in regard to the calculation with ∞. The results hold also for functions of n variables and for much more general spaces.

With the aid of the above defined saltus functions, we are thus able to tell various things—by no means obvious—about unconditioned functions. But, one may say, the properties describe the function $g(x)$ only indirectly; directly they describe the character of the associated saltus functions, which turn out to be of special character. Have we any non-trivial qualifications of unrestricted functions that describe them directly? Yes. But to state them we shall need certain additional notions.

It will be more convenient now to deal with real, single-valued functions $g(x)$ of two real variables, instead of one, but, as before, unconditioned as to continuity. And correspondingly the point sets we shall speak of will be planar.

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* Blumberg, Annals of Mathematics, (2), vol. 18 (1917), p. 147.

† Blumberg, loc. cit.

A (planar) point set N is said to be *nowhere-dense*, if in every circle of the plane there is another circle containing no points of N. For example, the set of points with integral cartesian coördinates is nowhere-dense.

A point set E is said to be *exhaustible*,* if it is the "sum" of a denumerable number of nowhere-dense sets, the term "sum" signifying here the set that is constituted by the elements that belong to one or more of the nowhere-dense sets. That is, an exhaustible set E is one expressible in the form

$$E=N_1+N_2+\cdots,$$

where N_i $(i=1, 2, \cdots)$ is a nowhere-dense set.

Not every set is exhaustible. The planar continuum, for example, is not exhaustible. Indeed, if E is any exhaustible set, every circle C contains points not belonging to E. For since N_1 is nowhere-dense, there is in C a circle C_1 containing no points of N_1; likewise, since N_2 is nowhere-dense, there is in C_1 a circle C_2 containing no points of N_2; and so on. In this manner we define a sequence of circles $C_1, C_2, \cdots$ such that C_i contains no points of $N_1, N_2, \cdots N_i$. Therefore, if P is a point contained in every circle C_i $(i=1, 2, \cdots)$, it belongs to none of the sets N_i and therefore not to E.

A denumerable set may be regarded as the sum of a denumerable number of sets, each consisting of one element. Therefore every denumerable set is exhaustible; and since the planar continuum is not exhaustible, it is not denumerable. The proof, by means of the notion of exhaustible sets, that the linear continuum is not denumerable is entirely analogous—or if we will, a corollary—so that we have another proof of the theorem of Cantor.

A *residual set* is the complement, with respect to the planar continuum, of an exhaustible set; in other words, if R is a residual set, the points of the continuum not belonging to R constitute an exhaustible set.

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* Cf. Denjoy, Journal de Mathématiques, (7), 1915, pp. 122–125.

The sum of two exhaustible sets $E=N_1+N_2+\cdots$ and $E'=N_1'+N_2'+\cdots$ is again representable as the sum $N_1+N_1'+N_2+N_2'+\cdots$ of a denumerable number of nowhere-dense sets, and is therefore exhaustible. On the other hand, the sum of a residual set and its complementary exhaustible set is the entire continuum, which is not exhaustible. A residual set cannot therefore be also exhaustible.

A set S is said to be an *I-region* (=open set), if every element of S is an inner point of S; i. e., no point of S is the limit of a sequence of points not in S. Thus the set consisting of the interior points of one or more polygons is an I-region; but the set consisting of the interior and the boundary points of a polygon is not an I-region, because a boundary point is the limit of a sequence of points outside of S.

A set S is said to be a *partial neighborhood of the point P*, if (a) S is an I-region, and (b) P is an interior or boundary point of S. The interior of a sector of a circle is, for example, a partial neighborhood of the center of the circle.

Suppose now that $g(x, y)$ is a one-valued, real function. If S is a set in the xy-plane, we shall understand by S' the spatial set consisting of the surface points of the surface $z=g(x, y)$ that correspond to the points of S; that is, as the point (x, y) varies over S, the point $(x, y, g(x, y))$ varies over S'. We shall say that the function $g(x, y)$ is *densely approached* at the point $A\equiv(x, y)$ of the xy-plane—or, in other words, that the point $A'\equiv(x, y, z)$ of the surface $z=g(x, y)$ is "densely approached"—if for every partial neighborhood S of A the point A' is a limit point of the set S'. Thus, if A' is densely approached, there exists in every partial neighborhood S of A a sequence of points $A_n=(x_n, y_n)$ approaching A as a limit such that the corresponding surface points $A_n'\equiv(x_n, y_n, g(x_n, y_n))$ approach A' as a limit.

An alternative definition of dense approach is the following: The function $g(x, y)$ is densely approached at $A\equiv(x, y)$, if for every positive number ϵ, there exists in the xy-plane a circle C with A as center such that the points (ξ, η) of C where $|g(\xi, \eta)-g(x, y)|<\epsilon$ are everywhere dense in C; i. e., every circle interior to C contains at least one point (ξ, η) such that $|g(\xi, \eta)-g(x, y)| < \epsilon$.

In terms of the defined notions, we may now state the following general property of real functions:

*For every real, one-valued function $g(x, y)$, the points of the xy-plane at which $g(x, y)$ is densely approached constitute a residual set. Conversely, if R is a residual set of the xy-plane, a function $g(x, y)$ exists that is densely approached at and only at the point of R.**

Without the explicit use of the notion of dense approach this theorem, apart from the converse portion, may be restated as follows:

With every real, one-valued function $g(x, y)$, there is associated a residual set R (dependent on g) of the xy-plane, such that if A is a point of R, N a partial neighborhood of A, and S a sphere having A' as interior point, there is at least one surface point $(x, y, g(x, y))$ in S such that (x, y) is in RN (the set common to R and N).

This property exhibits a remarkable degree of *regularity* possessed by every real function. The property is stated in so-called *descriptive* terms, involving merely the notion of density (nowhere-dense, everywhere dense, exhaustible, residual), but not that of length or measure. In such "descriptive" considerations, it is the residual set that plays the rôle of one rich in elements, while the exhaustible set manifests itself as relatively negligible. Our property asserts, then, that if we neglect a certain exhaustible set of the xy-plane, every remaining point (x, y) will be one of dense approach—one, therefore, such that the surface points of $z=g(x, y)$ cluster about (x, y, z) with a large degree of what we may roughly think of as *descriptive symmetry*.

The following property shows that the degree of *symmetrical clustering* is even more considerable:

With every real, one-valued function $g(x, y)$, there is associated a residual set R of the xy-plane, such that if A is a

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* Blumberg, Transactions of this Society, vol. 24 (1922), p. 113.

*point of R, A' the corresponding point of the surface $z=g(x, y)$, S a sphere with A' as center, N a partial neighborhood of A, and K the set of points of N that are projections of points of S that belong to the surface $z=g(x, y)$, then K is not exhaustible, and furthermore, K is everywhere dense in the portion of N that lies in a circle of the xy-plane having A as center and a sufficiently small radius (which may vary with A).**

The following property shows that, no matter how discontinuous a function may be, it yet possesses a certain remarkable degree of smoothness:

With every one-valued, real function $g(x, y)$, there may be associated—not uniquely of course—an everywhere dense set D of the xy-plane (i. e., one containing at least one point in every circle of the plane) such that $g(x, y)$ is continuous if (x, y) ranges over D.†

Although the first two properties for $g(x, y)$ were formulated on the assumption that $g(x, y)$ is one-valued, they hold essentially for many-valued functions, as we mentioned earlier. We state the extended theorem for the first property:

Let $g(x, y)$ be any real function defined for the entire xy-plane and taking at every point at least one value; the number of values may change, however, from point to point and vary from 1 to c, the cardinal number of the continuum. Then the points (x, y) such that every surface point $(x, y, g(x, y))$ is densely approached by the surface $z=g(x, y)$ constitute a residual set of the xy-plane.

Our properties, while stated for functions of two variables, hold also for functions of one or of any number of variables. These properties are, moreover, capable of wider extension. For example, it is not necessary to insist that $g(x, y)$ shall be defined at every point of the xy-plane. We are thus led to functions defined in more general domains than that of euclidean n-space. It turns out upon examination that it is

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* Blumberg, TRANSACTIONS, loc. cit

† Blumberg, loc. cit. The continuity of $g(x, y)$ over D means that if (x, y) is a point of D and $\{(x_n, y_n)\}$ a sequence of points of D having (x, y) as limit, then $\lim g(x_n, y_n)=g(x, y)$.

necessary to draw upon a relatively small number of the properties of the plane in order to build the proofs. It is these properties that lead logically to our descriptive theorems; we may, therefore, instead of starting with a definite spatial object, formulate a set of properties as postulates sufficient to insure our theorems as consequences, and thus arrive at an abstract generalized theory. Without entering upon extreme refinements of generalization, we may say that our theorems hold if $g(P)$, instead of being a function of a point $P=(x, y)$, is a real, one-valued function of an element P that ranges over any set $\mathfrak{S}$ whatsoever, instead of the plane; in other words, with every element P of $\mathfrak{S}$ there is associated a real number $g(P)$. $\mathfrak{S}$ is furthermore subject to the following four conditions:

1. $\mathfrak{S}$ is *metric*;* that is to say, with every pair of elements P and Q of $\mathfrak{S}$ there may be associated a non-negative, real number, which we denote by $\overline{PQ}$ (Fréchet's *écart*) in such a way that if P, Q, and R are any three elements of $\mathfrak{S}$, then

$$\text{(a)} \qquad \overline{PQ} = \overline{QP},$$

$$\text{(b)} \qquad \overline{PP} = 0,$$

$$\text{(c)} \qquad \overline{PQ} + \overline{QR} \geqq \overline{PR}$$

If, in particular, $\mathfrak{S}$ is the set of points in n-dimensional space, the number $\overline{PQ}$ may be taken as the distance between P and Q, and the conditions (a), (b), and (c) obviously hold. If $\mathfrak{S}$ is the set of continuous curves $y=g(x)$, $0 \leqq x \leqq 1$, and P and Q stand for two such curves $y=g_1(x)$ and $y=g_2(x)$, then $\overline{PQ}$ may be defined as max $|g_1(x)-g_2(x)|$, $0 \leqq x \leqq 1$; this definition renders (a), (b), and (c) valid.

2. $\mathfrak{S}$ is a "*complete space*" (vollständiger Raum);† that is to say, if $\{P_1, P_2, \cdots, P_n, \cdots\}$ is a *regular*‡ sequence of

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* Cf. for example, Fréchet, RENDICONTI DI PALERMO, vol. 22 (1907), p. 1, and Hausdorff, *Grundzüge der Mengenlehre*, 1914, p. 211.

† Hausdorff, loc. cit., p. 315.

‡ The sequence $\{P_1, P_2, \cdots, P_n, \cdots\}$ is *regular*, if for every positive there exists an integer n_ϵ such that $\overline{P_\lambda P_\mu} < \epsilon$ for $\lambda > n_\epsilon$ and $\mu > n_\epsilon$.

elements of $\mathfrak{S}$, there exists a limit element P of $\mathfrak{S}$ (i. e., an element P with the property $\lim_{n\to\infty} \overline{P_n P} = 0$).

If, for example, $\mathfrak{S}$ is taken to be the set of rational numbers of the linear continuum, it is not complete, since a regular sequence of rational numbers may have an irrational limit, i. e., a limit not belonging to $\mathfrak{S}$.

3. $\mathfrak{S}$ contains a denumerable subset T that is dense in $\mathfrak{S}$. This means that if P is an element of $\mathfrak{S}$, k a positive number, and S the ensemble of points Q of $\mathfrak{S}$ such that $\overline{PQ} < k$ (S may then be regarded as the set of points interior to the sphere having P as center and k as radius) then at least one point of T belongs to S. In the case of the plane, T may be chosen as the set of points both of whose coördinates are rational.

4. $\mathfrak{S}$ has no isolated points. That is to say, if P is a point of $\mathfrak{S}$ and S the sphere of center P and radius k defined above, then S contains at least one point of $\mathfrak{S}$ different from P.

As particular examples of a complete, metric space with a dense denumerable subset and without isolated points, we may mention the following:

(a) *Ordinary euclidean n-space.*

(b) *Hilbert space*, which is the set of those points in the space of a denumerable number of dimensions that are at a finite distance from the origin; in other words, the set of infinite sequences $\{x_1, x_2, \cdots\}$ such that $x_1^2 + x_2^2 + \cdots$, the squared distance of the point $(x_1, x_2, \cdots)$ from the origin, is finite. The écart between two elements $P = (x_1, x_2, \cdots)$ and $Q = (y_1, y_2, \cdots)$ is defined as $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots}$.

(c) *Function space*, which consists of all real, continuous functions $g(x)$ defined for $a \leqq x \leqq b$. The écart between two points, i. e., two curves, $y = g_1(x)$ and $y = g_2(x)$, is defined as $\max |g_1(x) - g_2(x)|$, $a \leqq x \leqq b$.

We now turn from the "descriptive" properties of functions i. e., those concerned essentially with density, to metric considerations, which involve lengths and areas.

The definition of a planar set of zero measure is analogous to that of a linear set of zero measure, which, we recall, is one enclosable in a denumerable number of intervals of arbitrarily small total length. To cover a planar set we use rectangles instead of intervals, so that a set is of measure zero, if it is enclosable in a denumerable number of rectangles of arbitrarily small total area.

Let S be a partial neighborhood of the point P; C_r, a circle of radius r having P as center; and S_r, the portion of S lying in C_r; then if

$$\lim_{r\to 0} \frac{\text{area } S_r}{\text{area } C_r}=0 ,$$

the set S is said to be a *vanishing* partial neighborhood of P. If, however, this limit does not exist, or if it exists and is different from 0, S is said to be a *non-vanishing* partial neighborhood of P. For instance, a sector of a circle is a non-vanishing partial neighborhood of the center; while the area bounded by an arc of a circle and the two tangents at its extremities is a vanishing partial neighborhood of the extremities. These are simple examples; more complicated ones indicating better the diversity of possibilities may be constructed by means of I-regions consisting of a denumerable number of simple areas like polygons or circles.

We shall say that the function $g(x, y)$ is *negligibly approached at the point* $P\equiv(x, y)$ *via the partial neighborhood* S, if a sphere exists with $P'\equiv(x, y, g(x, y))$ as center such that if T' is the set of points of the surface $z=g(x, y)$ that lie in the sphere, and T is the projection of T' upon the xy-plane, then the portion of T that lies in S is of measure zero.

We may now state the following theorem:*

Let $g(x, y)$ *be a real, single-valued function defined for the entire xy-plane. Then the set of points of the xy-plane that possess a non-vanishing partial neighborhood via which* g *is negligibly approached is of zero measure.*

In other words, if we are blind to what happens at the points of a certain set Z of zero measure, there is visible at every point P of the surface $z=g(x, y)$ an appreciable clustering

* TRANSACTIONS, loc. cit.

of the points of the surface via every non-vanishing neighborhood of P, appreciable, in the sense of non-negligible approach. Again (and now from the point of view of measure) a remarkable degree of regularity.

We shall mention one more property, which reveals even more strikingly the smoothness inherent in every function.

To this end, we first extend our definition of measure. Let S be a point set contained in a rectangle of area a; and C, the complementary set with respect to the rectangle, i. e., the set of points of the rectangle not belonging to S. Then we say that S is of *exterior measure* k if (1) it is possible to cover S by means of a denumerable number of rectangles of total area exceeding k by as little as we please; and (2) it is impossible to cover S by means of a denumerable number of rectangles of total area less than k. Similarly, we define the exterior measure m of the complementary set. These numbers k and m always exist. Not so, however, with the *measure* of S; it may or may not exist. For it may happen, as examples have shown, that $k+m>a$; in this case, it is not possible to assign a measure to S that may, without involving essential contradictions, be regarded as a generalized length. On the other hand, if $k+m$ comes out equal to a we define k as the measure of S. In case the exterior measure is zero, the measure always exists; this is why it was not necessary, in connection with sets of zero measure, to refer to the possibility of non-measurable sets.

Let now ϵ be a positive number; $P\equiv(\xi, \eta)$, a point of the xy-plane; and $M_{P\epsilon}$, the set of points (x, y) such that $|g(x,y)-g(\xi, \eta)|<\epsilon$. Let C be a circle of the xy-plane having P as center and r as radius; and $M_{P\epsilon C}$, the subset of $M_{P\epsilon}$ that lies in the circle C. Then if

$$\lim_{r\to 0}\frac{\text{exterior measure of } M_{P\epsilon C}}{\text{area of } C}=1 ,$$

the set $M_{P\epsilon}$ is said to have the *exterior metric density* 1 at the point P. In this case—and from the point of view of regarding, for the nonce, the exterior measure as representing the metric extent of a set—the set M_{P_ϵ} has the maximum possible *metrical clustering* about the point P; and in case the exterior metric density does not exist or is different from 1, the metrical clustering is deficient. Now it may happen that the function $g(x, y)$ and the point P are such that for every $\epsilon > 0$ the set M_{P_ϵ} has 1 as its exterior metric density at P. In this case, we say that $g(x, y)$ is *quasi-continuous* at P—the term *quasi* referring to our use of exterior measure instead of measure.*

We may now state the following property :†

The function g is quasi-continuous except at the points of a set of zero measure.

The metric properties of unconditioned functions may also be extended to many-valued functions. While they hold for all euclidean spaces of a finite number of dimensions, their extension to a space of a denumerable number of dimensions would require a satisfactory definition of measure for such a space; this is, at present, wanting. A general space can, of course, be defined by means of postulates for which the metric theorems hold, but these postulates would be more stringent than in the case of the descriptive theorems, where comparatively few and simple conditions were found adequate.

In closing, I shall mention several properties of a third type concerning unconditioned functions. As before, these properties hold for single-valued functionsg $(x_1, x_2, \cdots, x_n)$ of n variables defined at every point of n-space.

‡*If S is the set of points where a givenfunction g $(x_1, x_2, \cdots, x_n)$ is continuous, there exists an infinite sequence $I_1, I_2, \cdots$, of I-regions such that S is the aggregate of points common to them*

* In case measure were used, we could properly regard $g(x, y)$ as essentially continuous, from the point view of the convention that sets the zero measure may be regarded as negligible.

† TRANSACTIONS, loc. cit.

‡ For the case of one variable, see W. H. Young, *Über die Einteilung der unstetigen Funktionen und die Verteilung ihrer Stetigkeitspunkte*, WIENER SITZUNGSBERICHTE, vol. 112, Abt. IIa, p. 1307; for the general case, H. Blumberg, *On the characterization of the s t of points of λ-continuity*, ANNALS OF MATHEMATICS, (2), vol. 25 (1923), p. 118.

all. Conversely, if S is representable as the common part of a sequence of I-regions, there exists a function $g(x_1, x_2, \cdots, x_n)$ *such that g is continuous at the points of S and discontinuous elsewhere.*

With the aid of the various types of saltus, we may define other types of continuity. A function is continuous in the ordinary sense, if its ordinary saltus is zero. In similar manner, we say that a function is *f-continuous, d-continuous, e-continuous,* or z-continuous* at a point P if its f-saltus, d-saltus, e-saltus, or z-saltus respectively vanishes at P. And we have the following theorem,† in which we may substitute for the letter λ any one of the four letters f, d, e, or z:

The set of points where a function of n variables is λ-continuous is representable as the common part of an infinite sequence of I-regions; conversely, if S is representable as the common part of a sequence of I-regions, a function exists that is λ-continuous at every point of S and λ-discontinuous elsewhere.‡

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* The e-saltus arises when exhaustible sets are regarded as negligible.

† Blumberg, ANNALS, loc. cit.

‡ The following articles,—all in the FUNDAMENTA MATHEMATICAE,—which the writer had occasion to see during the course of publication of the present address, belong in whole or in part to our subject: W. Sierpiński, *Sur une généralisation de la notion de la continuité approximative,* vol. 4 (1923), p. 124; A. Rajchman and S. Saks, *Sur la dérivabilité des fonctions monotones,* loc. cit., p. 204; S. Saks, *Sur les nombres dérivés des fonctions,* vol. 5 (1924), p. 98; S. Kempisty, *Sur les fonctions approximativement discontinues,* vol. 6 (1924), p. 6.