A TRIVIAL TAUBERIAN THEOREM.*

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The name Tauberian was introduced by Hardy† to describe a very interesting type of theorem in connection with summable series; he and others have enunciated a considerable number of such theorems bearing on various specific definitions of summability. We may indicate the general character of a Tauberian theorem as follows. The ordinary questions on summability consider two related sequences (or other functions) and ask whether it will be true that one sequence possesses a limit whenever the other possesses a limit, the limits being the same; a Tauberian theorem appears, on the other hand, only if this is untrue, and then asserts that the one sequence possesses a limit provided the other sequence both possesses a limit and satisfies some additional condition restricting its rate of increase. The interest of a Tauberian theorem lies particularly in the character of this additional condition, which takes different forms in different cases. Thus, if the condition is to be imposed on the term u_n of a series, it may take any of the forms (for which I write alternative notations)

$$|u_n| < Kf_n : \quad u_n = O(f_n) ,$$
$$u_n < Kf_n : \quad u_n = O_+(f_n) ,$$
$$u_n > -Kf_n : \quad u_n = O_-(f_n) ,$$
$$u_n/f_n \to 0 : \quad u_n = o(f_n) ,$$
$$\limsup_{n\to\infty} u_n/f_n = 0 : \quad u_n = o_+(f_n) ,$$
$$\liminf_{n\to\infty} u_n/f_n = 0 : \quad u_n = o_-(f_n) ;$$

* Presented to the Society, December 30, 1924.

† With respect to the name, see Hardy and Littlewood, PROCEEDINGS OF THE LONDON SOCIETY, vol. 2, (1912-13), p. 1.

in each case K is a positive constant and (f_n) a given sequence of positive elements.

It has seemed to me worth while to discuss a case so trivial that the reasoning is transparent, but involving a parameter in such a way that for various values of the parameter it exhibits several forms of the condition, with assurance that the best form for each value has been chosen.*

Consider then the series

$$u_1+u_2+u_3+\cdots;$$

let

$$x_n=u_1+u_2+\cdots+u_n.$$

Apply the transformation

$$(A)\qquad y_1=\frac{x_1}{1-\alpha};\qquad y_n=\frac{x_n-\alpha x_{n-1}}{1-\alpha},\qquad n>1,$$

where α is any real number other than 1. It is at once obvious that (A) is regular: whenever x_n has a limit, y_n has the same limit.†

We now inquire whether the existence of a limit for y_n implies the existence of a limit for x_n. For the inverse transformation we find

$$(A^{-1})\qquad x_n=(1-\alpha)\left[y_n+\alpha y_{n-1}+\alpha^2 y_{n-2}+\cdots+\alpha^{n-1}y_1\right].$$

This is of the form $x_n=\sum a_{n,k}y_k$, with

$$a_{n,k}=(1-\alpha)\alpha^{n-k},\ k\leqq n;\ a_{n,k}=0,\ k>n.$$

Thus, by the Silverman-Toeplitz Theorem,‡ if A^{-1} is to be regular, $|\alpha|<1$; and if $|\alpha|<1$, since

$$\sum_{k=1}^{n}a_{n,k}=1-\alpha^n\to 1\ ;\ \sum_{k=1}^{n}|a_{n,k}|=\frac{|1-\alpha|}{1-|\alpha|}\left[1-|\alpha|^n\right]<\frac{|1-\alpha|}{1-|\alpha|},$$

* A recent paper by R. Schmidt, MATHEMATISCHE ZEITSCHRIFT, vol. 22 (1925), pp. 89–152, for the first time undertakes a systematic general study of Tauberian theorems. Schmidt's work yields as special cases some, but not all, of the results of this note.

† This holds also if α is complex.

‡ See the author's *Report on topics in the theory of divergent series*, this BULLETIN, vol. 28 (1922), p. 19.

by the same theorem, A^{-1} will be regular. In case, then, $|\alpha|<1$, A is equivalent to convergence; no Tauberian condition is required to insure that if y_n has a limit, x_n shall have the same limit.

Suppose now $|\alpha|>1$. Write

$$\lambda = (1-\alpha)\left[\frac{y_1}{\alpha}+\frac{y_2}{\alpha^2}+\cdots\right];$$

the series converges, since y_n is bounded. We find

$$x_n-\lambda\alpha^n = (\alpha-1)\left[\frac{y_{n+1}}{\alpha}+\frac{y_{n+2}}{\alpha^2}+\cdots\right].$$

The right-hand side of this equation is in the form $\sum b_{n,k}\,y_k$, where

$$b_{n,k}=0\,, \quad k\leqq n\,; \quad b_{n,k}=\frac{\alpha-1}{\alpha^{k-n}}\,, \quad k>n\,.$$

Hence

$$\lim_{n\to\infty} b_{n,k}=0\,; \quad \sum_{k=1}^{\infty} b_{n,k}=1\,; \quad \sum_{k=1}^{\infty}|b_{n,k}|=\frac{|\alpha-1|}{|\alpha|-1}\,;$$

and by the Hildebrandt-Carmichael generalization* of the Silverman-Toeplitz theorem, the transformation $||b_{n,k}||$ is regular. Since $y_n\to l$,

$$x_n-\lambda\alpha^n\to l\,,$$

and

$$u_n-\lambda\alpha^{n-1}(\alpha-1)\to 0\,.$$

We now consider separately the cases $\alpha>1$, $\alpha<-1$.

Let $\alpha>1$. Then if $\lambda\neq 0$, $x_n\to(\operatorname{sgn}\lambda)\infty$, $u_n\to(\operatorname{sgn}\lambda)\infty$; if $\lambda=0$, $x_n\to l$, $u_n\to 0$. In order to insure by a condition on u_n that $x_n\to l$ we must prevent u_n from becoming definitely (i. e., with one sign) infinite. This is secured by demanding that u_n be bounded *both above and below*:

$$|u_n|<K: \quad u_n=O(1)\,.$$

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* See the author's *Report*, loc. cit., p. 20; also Hildebrandt, this BULLETIN, vol. 29 (1923), p. 314.

Let $\alpha < -1$. Then if $\lambda \neq 0$, x_n and u_n each oscillate between $+\infty$ and $-\infty$; if $\lambda = 0$, $x_n \to l$, $u_n \to 0$. In order to insure that $x_n \to l$, it suffices to bound u_n *in one direction only*; that is, to impose *one* of the conditions

$$u_n < K : \qquad u_n = O_+(1) \; ;$$

$$u_n > -K : \qquad u_n = O_-(1) \; .$$

There remains the case $\alpha = -1$. Here

$$(1) \qquad y_n = \frac{x_n + x_{n-1}}{2} , \qquad n > 1 \; .$$

Neither of the preceding methods applies. The failure, furthermore, lies not merely in the proof, but in the facts. Conditions of the forms O, O_+, O_- are not sufficient; it is possible for y_n to possess a limit while x_n and u_n oscillate finitely or infinitely. For instance, let

$$u_1 = l + a : \quad u_n = (-1)^{n-1} 2a \; , \quad n > 1 \; ;$$

where $a > 0$. Then

$$x_n = l + (-1)^{n-1} a \; ;$$

$$y_1 = \frac{l+a}{2} \; ; \quad y_n = l \; , \quad n > 1 \; .$$

If we call lim sup $x_n = \bar{x}$, lim inf $x_n = \underline{x}$ (whether finite or infinite), with similar notations for u_n,

$$y_n \to l \; ; \quad \bar{x} = l + a \; , \quad \underline{x} = l - a \; ; \quad \bar{u} = 2a \; , \quad \underline{u} = -2a \; .$$

Again, let

$$y_n = \frac{(-1)^{n-1}}{n} \; ;$$

then

$$x_n = (-1)^{n-1} 2 \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \; ;$$

$$u_n (= -1)^{n-1} \left[-\frac{2}{n} + 4 \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \right] \; ;$$

so that

$$y_n \to 0\ ;\ \bar{x}=+\infty\ ;\ \underline{x}=-\infty\ ;\ \bar{u}=+\infty\ ,\ \underline{u}=-\infty\ .$$

To deduce an appropriate condition in this case, note that

$$(2) \qquad y_n=x_n-\tfrac{1}{2}u_n\ ; \quad y_n=x_{n-1}+\tfrac{1}{2}u_n\ , \quad n>1\ .$$

From (1) and (2), we have

$$\bar{x}=2l-\underline{x}\ ;$$
$$\bar{x}=l+\tfrac{1}{2}\bar{u}\ , \quad \bar{x}=l-\tfrac{1}{2}\underline{u}\ ;$$
$$\underline{x}=l+\tfrac{1}{2}\underline{u}\ , \quad \underline{x}=l-\tfrac{1}{2}\bar{u}\ .$$

Thus if any one of the four limits $\bar{x}$, $\underline{x}$, $\bar{u}$, $\underline{u}$ is finite, all are finite and $\bar{u}=-\underline{u}$. If none are finite, then $\bar{x}=\bar{u}=+\infty$, $\underline{x}=\underline{u}=-\infty$. In order to be able to assert that $x_n \to l$, we must have $\bar{x}=\underline{x}=l$, hence $\bar{u}=\underline{u}=0$. But if *either* $\bar{u}=0$ or $\underline{u}=0$, it follows that $\bar{x}=l$, $\underline{x}=l$, and hence $x_n \to l$. Therefore it suffices in this case to impose one of the conditions

$$\limsup_{n\to\infty} u_n=0\ :\quad u_n=o_+(1)\ ;$$

$$\liminf_{n\to\infty} u_n=0\ :\quad u_n=o_-(1)\ .$$

The results may be collected as follows:

For the transformation A, if y_n has the limit l, x_n will also have the limit l:

in case $\alpha<-1$, if $u_n=O_+(1)$ or $u_n=O_-(1)$;
in case $\alpha=-1$, if $u_n=o_+(1)$ or $u_n=o_-(1)$;
in case $-1<\alpha<1$, without any Tauberian condition;
in case $\alpha>1$, if $u_n=O(1)$.

Thus the cases of bilateral O-condition, unilateral O-condition, and absence of Tauberian condition occur for whole intervals of parameter values, separated by isolated points which are characterized by o-conditions or by the breaking down of the transformation. It is also of interest that for $\alpha>1$, the form of condition given may be replaced by the formally weaker condition,

$$\liminf_{n\to\infty} u_n \neq +\infty \quad \textit{and} \quad \limsup_{n\to\infty} u_n \neq -\infty\ ;$$

and for $\alpha = -1$, the condition given may be replaced by

$$\lim_{n\to\infty} u_n \textit{ exists.}$$

Extensions can be made by allowing y_n to involve more than two consecutive x's. If for example

$$y_n = \frac{x_n + \alpha x_{n-1} + \beta x_{n-2}}{1+\alpha+\beta},$$

the different conditions depend on the location of the roots (real or complex) of the polynomial $z^2 + \alpha z + \beta$ with respect to the unit circle $|z| = 1$. No new kinds of result appear, and the analysis is less transparent; as I consider the Tauberian theorem of this note to have no intrinsic importance and to be of interest merely as an indication of general relationship of Tauberian conditions to one another, I omit the discussion of the extensions mentioned.

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