SOME MEAN-VALUE THEOREMS CONNECTED WITH COTES'S METHOD OF MECHANICAL QUADRATURE

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1. *Introduction.* It is the purpose of this paper to prove several mean-value theorems whose importance in the problem of mechanical quadrature will appear later.

2. *Extension of a Theorem of Birkhoff.* The following theorem has been proved by G. D. Birkhoff* for the cases $n = 1, 2, 3, 4, 5$. The proof was made by computations based on tables computed by Cotes. The method is not applicable to the general case.

THEOREM I. *A function $u(x)$ is continuous with its first $2n+2$ derivatives in an interval (a, b), and its $(2n+2)$th derivative itself possesses a derivative for every value of x between a and b. If the function vanishes at the points a and b, and if its first derivative vanishes at $2n+1$ points of the interval, equally spaced and including the end points a and b, then the $(2n+3)$th derivative of the function vanishes between a and b.*

By a successive application of Rolle's theorem it may be inferred directly that $u^{(2n+1)}(x)$ must vanish in the interval (a, b), and that without use of the hypothesis that the points in question are equally spaced. It is important to note, however, that if the points are not equally spaced the conclusion of the theorem is not valid. This fact is made clear by a simple example:

$$u(x) = (x^2-4)^2(4x-1).$$

Here

$$u(2) = u(-2) = u'(2) = u'(-2) = u'(1) = 0.$$

Yet the fifth derivative of $u(x)$ is a constant not zero.

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* G. D. Birkhoff, *General mean-value and remainder theorems*, TRANSACTIONS OF THIS SOCIETY, vol. 7 (1906), p. 131.

Suppose that the $2n+1$ equally spaced points have coordinates $x_{-n}, x_{-n+1}, \cdots, x_0, x_1, \cdots, x_n,$ and suppose the points arranged from left to right on the line in order of increasing indices.

Then

$$x_{-n} = a, \quad x_n = b.$$

We first make a linear transformation of the independent variable transforming the points in question into the points $-n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n$. Clearly, if the theorem is valid for this set of points, it is valid for the original set. We may now base our proof on a lemma proved by J. F. Steffensen.*

LEMMA. *If m and n are any positive integers, then the function*

$$(1) \qquad Q(x) = \int_{-m}^{x} x(x^2-1)(x^2-2^2)\cdots(x^2-n^2)\,dx$$

does not vanish in the interval $-m < x < m$.

The proof may readily be supplied by geometrical considerations. We may state our hypotheses on $u(x)$ as follows:

$$u(-n) = u(n) = u'(k) = 0,$$
$$k = -n, \ -n+1, \ \cdots, \ -1, 0, 1, \ \cdots, \ n-1, \ n.$$

Since $u(-n) = 0$,

$$u(x) = \int_{-n}^{x} u'(x)\,dx.$$

Now $u'(x)$ vanishes at the same points as the function

$$P(x) = x(x^2-1)(x^2-2^2)\cdots(x^2-n^2).$$

Determine a function $R(x)$ by the equation

$$u'(x) = P(x)R(x),$$

* J. F. Steffensen, CONFÉRENCES DU CINQUIÈME CONGRÈS DES MATHÉMATICIENS SCANDINAVES, p. 126.

$R(x)$ being defined at an integer r by the equation

$$R(r) = \lim_{x \to r} \frac{u'(x)}{P(x)}.$$

Integrating this equation from $-n$ to n, and recalling that $u(-n) = u(n) = 0$, we have

$$u(n) - u(-n) = \int_{-n}^{n} P(x) R(x) \, dx = 0.$$

On integrating the right hand side of this equation by parts, and setting

$$Q(x) = \int_{-n}^{x} P(x) \, dx,$$

it follows that

$$0 = Q(x) R(x) \Big|_{-n}^{n} - \int_{-n}^{n} R'(x) Q(x) \, dx = \int_{-n}^{n} R'(x) Q(x) \, dx.$$

Now by the lemma, $Q(x)$ is a function of one sign in the interval $-n \leqq x \leqq n$, so that we may apply the first mean-value theorem for integrals. We have thus

$$R'(\zeta) \int_{-n}^{n} Q(x) \, dx = 0, \qquad -n < \zeta < n.$$

Now

$$\int_{-n}^{n} Q(x) \, dx \neq 0,$$

since $Q(x)$ does not change sign in the interval $(-n, n)$ and is not identically zero. Hence it follows that

$$R'(\zeta) = 0. \tag{2}$$

We may now make use of Rolle's theorem and of equation (2) to obtain the result desired through the following device. Form the function

$$\Phi(x) = P(x) [R(x) - R(\zeta)].$$

On account of the relation (2), it is seen that the quantity in brackets vanishes at least twice in the point $x = \zeta$. $P(x)$ vanishes $2n+1$ times in the interval $-n \leqq x \leqq n$. Hence, allowing for multiple zeros, we see that $\Phi(x)$ vanishes at least $2n+3$ times in that interval, whether or not ζ is an integer (that is, a point at which $P(x)$ vanishes). Now by successive applications of Rolle's theorem* we see that $\Phi^{(2n+2)}(x)$ vanishes at least once in the interval $(-n, n)$.

Since the $(2n+2)$th derivative of $P(x)$ is identically zero, and since $P(x)\cdot R(x)$ is $u'(x)$, it follows that

$$\Phi^{(2n+2)}(\xi) = u^{(2n+3)}(\xi) = 0, \qquad -n<\xi<n.$$

The theorem is thus established.

3. *A Companion Theorem.* The second theorem is similar in character to the first, but deals with an even number of points where the first theorem dealt with an odd number. The conclusion, however, is essentially different in that the order of the highest derivative whose vanishing can be inferred is one less than the number of conditions imposed on $u(x)$, while in Theorem I these numbers were equal.†

THEOREM II. *A function $u(x)$ is continuous with its first $2n$ derivatives in an interval (a, b), and its $2n$th derivative itself possesses a derivative for every value of x between a and b. If the function vanishes at a and b, and if its first derivative vanishes at $2n$ points of the interval, equally spaced and including the end points a and b, then the $(2n+1)$th derivative of the function vanishes between a and b.*

No loss of generality will be incurred by taking the points as $-2n+1, -2n+3, \cdots, -3, -1, 1, 3, \cdots, 2n-3, 2n-1$. Then $a=-2n+1$, $b=2n-1$. Since $u(a)=u(b)=0$, we have

$$\int_a^b u'(x)\,dx = u(b)-u(a) = 0.$$

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* See, for example, de la Vallée Poussin, *Cours d'Analyse,* vol. 1, p. 66 (5th edition).

† From the point of view of the general theory of mean-values, as set forth by Birkhoff, this is an essential distinction, Theorem II coming under the general case and Theorem I being an exceptional case.

Set

$$P(x) = (x^2-1)(x^2-3^2)\cdots(x^2-\overline{2n-1}^2),$$

and determine $R(x)$ by the equation

$$u'(x) = P(x)R(x).$$

Then

$$\int_a^b P(x)R(x)\,dx = 0. \tag{3}$$

Now set

$$q(x) = \frac{P(x)}{x-2n+1}, \qquad Q(x) = \int_a^x q(x)\,dx.$$

We can show that $Q(x)$ is a function of one sign in the interval (a,b). By the lemma, $Q(x)$ is a function of one sign in the interval $(a, 2n-3)$. (The lemma is geometrical, and is applicable after a change of unit or a shift of origin.) $Q(x)$ vanishes at a and at $2n-3$* but is positive at intermediate points, $q(x)$ is positive in the interval $(2n-3, b)$, and consequently $Q(x)$ is never negative in the interval (a,b).

From equation (3) we have

$$\int_a^b P(x)\,R(x)\,dx = \int_a^b (x-2n+1)\,q(x)\,R(x)\,dx = 0.$$

Integrating by parts, we obtain

$$(x-2n+1)R(x)\,Q(x)\Big|_b^a - \int_a^b \frac{d}{dx}[(x-2n+1)R(x)]\,Q(x)\,dx = 0.$$

The first term on the left hand side drops, so that we have on applying the first mean-value theorem for integrals

$$\frac{d}{dx}\Big[(x-2n+1)R(x)\Big]_{x=\zeta}\cdot\int_a^b Q(x)\,dx = 0, \qquad a<\zeta<b.$$

Since $\int_a^b Q(x)\,dx \neq 0$, it follows that

$$\frac{d}{dx}\Big[(x-2n+1)R(x)\Big]_{x=\zeta} = 0. \tag{4}$$

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* This follows because $q(x-1) = q(-x-1)$.

We now employ a device similar to that used in the proof of Theorem I. Set

$$\begin{aligned}\Phi(x) &= P(x)R(x) - R(\zeta)(\zeta - 2n + 1)q(x)\\ &= q(x)[R(x)(x - 2n + 1) - R(\zeta)(\zeta - 2n + 1)].\end{aligned}$$

Now on account of the relation (4) it is seen that the quantity in brackets has at least two zeros in the point $x = \zeta$; and $q(x)$ has $(2n - 1)$ zeros. It follows then that $\Phi(x)$ has at least $(2n + 1)$ zeros in the closed interval (a, b). By Rolle's theorem $\Phi^{(2n)}(x)$ must vanish at least once in that interval. Since $P(x)R(x)$ is $u'(x)$, and since the $2n$th derivative of $q(x)$ is identically zero, it follows that

$$\Phi^{(2n)}(\xi) = u^{(2n+1)}(\xi) = 0, \qquad a < \xi < b.$$

The theorem is thus established.

4. *Applications to Mechanical Quadrature.* Let us now apply these theorems to the problem of mechanical quadrature. Suppose a function $f(x)$ is known at equally spaced points $x_1, x_2, x_3, \cdots x_{2n+1}$, and suppose it be required to find an approximate expression for

$$\int_{x_1}^{x_{2n+1}} f(x)\,dx.$$

Let $F(x)$ be the polynomial of degree $2n$ at most, taking on the known values of $f(x)$ at the given points. Then form the function

$$u(x) = \int_{x_1}^{x} [f(x) - F(x)]\,dx - A\int_{x_1}^{x} P(x)(x - x_{n+1})\,dx,$$

where

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_{2n+1}),$$

and where A is to be so determined that $u(x_{2n+1}) = 0$. This is possible since

$$\int_{x_1}^{x_{2n+1}} P(x)(x - x_{n+1})\,dx \neq 0.*$$

* This may be seen by writing

$$\int_{x_1}^{x_{2n+1}} P(x)(x - x_{n+1})\,dx = -\int_{x_1}^{x_{2n+1}} dx \int_{x_1}^{x} P(t)\,dt$$

and then applying the Lemma.

Now $u(x)$ satisfies all the conditions of Theorem I (making proper assumptions regarding the continuity of $f(x)$, and consequently

$$u^{(2n+3)}(\xi) = 0, \qquad x_1 < \xi < x_{2n+1}.$$

That is,

$$f^{(2n+2)}(\xi) - A(2n+2)! = 0, \quad A = f^{(2n+2)}(\xi)/(2n+2)!$$

Since $u(x_{2n+1}) = 0$, we have

$$(5) \quad \int_{x_1}^{x_{2n+1}} f(x)\,dx = \int_{x_1}^{x_{2n+1}} F(x)\,dx + \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\int_{x_1}^{x_{2n+1}} P(x)(x - x_{n+1})\,dx,$$

a remainder formula for mechanical quadrature.

In case the number of points is even, we use theorem II and obtain

$$(6) \quad \int_{x_1}^{x_{2n}} f(x)\,dx = \int_{x_1}^{x_{2n}} F(x)\,dx + \frac{f^{(2n)}(\xi)}{(2n)!}\int_{x_1}^{x_{2n}} (x - x_1)\cdots(x - x_{2n})\,dx.$$

These formulas were given by Birkhoff in the paper already cited, but were established only for small values of n. Steffensen established formula (5) by other methods in the article above cited.

5. *Extensions.* The methods here employed might be used to prove certain other mean-value theorems involving equally spaced points, and to which Rolle's theorem is not directly applicable. For example, I have proved the following theorem.

THEOREM III. *A function $u(x)$ having the same amount of continuity as that of Theorem* I *vanishes at the points a and b. If (a, b) is divided into $2n+2$ equal parts, and if $u'(x)$ vanishes at the interior points of division, then $u^{(2n+3)}(x)$ vanishes at an interior point of (a, b).*

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