function f^*. Hence their difference converges in the mean to zero.

This last theorem may be looked upon as giving a necessary and sufficient condition that two normal orthogonal sets $\{u_k\}$, $\{v_k\}$ be equivalent in the sense that the formal series for an arbitrary function in terms of them always converge in the mean to the same function.

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ON THE COMPLETE INDEPENDENCE OF THE FUNCTIONAL EQUATIONS OF INVOLUTION*

BY C. C. MAC DUFFEE

1. *Introduction.* The three fundamental theorems or "laws" of involution are commonly written

$$\text{I } a^m \cdot a^n = a^{m+n}, \quad \text{II } (a^m)^n = a^{mn}, \quad \text{III } a^n \cdot b^n = (ab)^n.$$

In § 2 of this paper these equations are abstractly formulated as functional equations. In § 3 it is proved that any function satisfying Equations I and II also satisfies III, provided certain underlying conditions A and B are fulfilled.

In § 4 a number system $\mathfrak{R}$ is introduced whose elements are the numbers $[\xi, r]$ where ξ and r are real numbers. Two operations, addition and multiplication, are introduced, and it is shown that these operations obey all the ordinary laws of algebra except the associative law of addition. This number system $\mathfrak{R}$ is then used in the discussion of the complete independence† of Equations I, II and III.

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† Consider n conditions. There are conceptually 2^n distinct cases to be considered according as Condition 1 holds or does not hold, Condition 2 holds or does not hold, $\cdots$, Condition n holds or does not hold. If none of these 2^n cases is empty, the n conditions are said to be completely independent. Cf. E. H. Moore, THE NEW HAVEN MATHEMATICAL COLLOQUIUM, (Yale University Press, 1910), pp. 81, 82

Subject to the conditions of § 2 the functional equations are not completely independent, for I and II imply III. However, the remaining 7 cases are non-empty, for in § 5 of this paper 7 functions are defined which obey the conditions of § 2, each fulfilling one of these 7 cases.

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2. *A General Formulation of the Functional Equations.* Consider a system Σ composed of the following terms:

(1) Two classes C, C' whose elements will be denoted respectively by c, c'.

(2) Three single-valued functions* or operations:

$$\times \text{ on } CC \text{ to } C, \quad \times' \text{ on } C'C' \text{ to } C', \quad +' \text{ on } C'C' \text{ to } C'.$$

(3) A single-valued function φ on CC' to C.

We assume that the following underlying conditions are satisfied:

A. The right-hand distributive law holds for C', namely

$$(c_1'+'c_2')\times' c_3' = (c_1'\times' c_3')+'(c_2'\times' c_3')$$

for every c_1', c_2', c_3'.

B. There is an element c_0 of C such that for every c there is a c' such that $c = \varphi(c_0, c')$.

We now consider the following postulates in connection with system Σ:

$$\text{I.} \qquad \varphi(c, c_1')\times\varphi(c, c_2') = \varphi(c, c_1'+'c_2'),$$

$$\text{II.} \qquad \varphi(\varphi(c, c_1'), c_2') = \varphi(c, c_1'\times' c_2'),$$

$$\text{III.} \qquad \varphi(c_1, c')\times\varphi(c_2, c') = \varphi(c_1\times c_2, c').$$

It is understood that these relations hold for all values of the arguments.

As an example of a system Σ in which Conditions A

* Thus for example, if c_1, c_2 are two elements of C, $\times(c_1, c_2)$ or $c_1\times c_2$ denotes a definite element c_3 of C.

and B and all three postulates are satisfied, we may identify the class C with the class of all real numbers $\geqq 1$, C' with the class of all numbers $\geqq 0$, $\times$ and $\times'$ with the ordinary operation of multiplication, $+'$ with ordinary addition, and $\varphi(c, c')$ with the power function $c^{c'}$.

3. THEOREM. *If system* Σ *satisfies Conditions* A *and* B, *and* φ *satisfies Postulates* I *and* II, *then* φ *also satisfies Postulate* III.

Suppose that c_0 is the element c_0 of Condition B, and that c_1 and c_2 are any two elements of C. Then by Condition B elements c_1' and c_2' exist so that

$$(1) \qquad c_1 = \varphi(c_0, c_1'), \qquad c_2 = \varphi(c_0, c_2').$$

Therefore we have

$$\begin{aligned} \varphi(c_1 \times c_2, c') &= \varphi(\varphi(c_0, c_1') \times \varphi(c_0, c_2'), c'), && (1), \\ &= \varphi(\varphi(c_0, c_1' +' c_2'), c'), && (\mathrm{I}), \\ &= \varphi(c_0, (c_1' +' c_2') \times' c'), && (\mathrm{II}), \\ &= \varphi(c_0, c_1' \times' c' +' c_2' \times' c'), && (A), \\ &= \varphi(c_0, c_1' \times' c') \times \varphi(c_0, c_2' \times' c'), && (\mathrm{I}), \\ &= \varphi(\varphi(c_0, c_1'), c') \times \varphi(\varphi(c_0, c_2'), c'), && (\mathrm{II}), \\ &= \varphi(c_1, c') \times \varphi(c_2, c'), && (1), \end{aligned}$$

which is precisely Postulate III.

4. *An Algebra Whose Elements are Pairs of Real Numbers.* Consider the class of all "numbers" or number pairs of the type $[\xi, r]$ where ξ and r are real numbers. The equality

$$[\xi_1, r_1] = [\xi_2, r_2]$$

implies that $r_1 = r_2$, and unless $r_1 = r_2 = 0$ it also implies that $\xi_1 = \xi_2$. The number $[\xi, 0]$ is supposed to be independent of ξ and will be called *zero*.

When $r_1 r_2 \neq 0$, addition of two such number pairs is defined by the identity

$$[\xi_1, r_1] + [\xi_2, r_2] = [\xi_3, r_3]$$

where r_3 and ξ_3 are defined by the relations

$$r_3 \sin \xi_3 = r_1 \sin \xi_1 + r_2 \sin \xi_2,$$
$$r_3 \cos \xi_3 = r_1 \cos \xi_1 + r_2 \cos \xi_2,$$
$$\frac{\xi_1 + \xi_2}{2} - \frac{\pi}{2} < \xi_3 \leqq \frac{\xi_1 + \xi_2}{2} + \frac{\pi}{2}.$$

When either r_1 or r_2 or both are zero, addition is defined by the relations

$$[\xi_1, 0] + [\xi_2, r_2] = [\xi_2, r_2], \qquad [\xi_1, r_1] + [\xi_2, 0] = [\xi_1, r_1].$$

The sum always exists uniquely.

The operation of multiplication is defined by the identity

$$[\xi_1, r_1] \times [\xi_2, r_2] = [\xi_1 + \xi_2, r_1 r_2].$$

The product always exists uniquely.

The algebra of such number pairs subject to the operations of addition and multiplication as defined above we shall call the algebra $\mathfrak{R}$. It is evident that both addition and multiplication in $\mathfrak{R}$ are commutative, for the identities which define these operations are symmetric in ξ_1, r_1 and ξ_2, r_2. It may be shown directly that multiplication is associative and distributive with respect to addition. It is worthy of note however that addition is not always associative. This fact is sufficient to show that $\mathfrak{R}$ is not isomorphic with any linear algebra.

5. *Concerning the Complete Independence of the Postulates.* In connection with three conditions 1, 2, 3 there are conceptually eight distinct cases to be considered, each case being defined by the holding or non-holding of each condition. If none of these cases is empty, the three conditions are completely independent. We wish to consider the complete independence of Postulates I, II, III of § 2 in a system Σ with Conditions A and B holding. To settle this question completely we must consider eight cases, and the existence of eight functions φ_{ijk} where i is 1 or 0 according as the function φ_{ijk} satisfies or does not satisfy I, j is 1 or 0 according as it satisfies or does not satisfy II, and k is 1 or 0 according as it satisfies or does not satis-

fy III. In view of the theorem of § 3 no function φ_{110} can exist, for every function φ in Σ satisfying A, B, I and II must necessarily satisfy III. We show that no further dependence theorems are possible by exhibiting examples of functions in each of the remaining seven cases.

In each of the following examples we shall use $\mathfrak{R}$ as the class C', and $\mathfrak{R}$ or a subclass of $\mathfrak{R}$ which is closed under the $\times$-operation as the class C. We shall denote by C_1 the subclass of $\mathfrak{R}$ composed of all numbers $[\xi, r]$ for which $r > 0$; by C_2 the subclass of $\mathfrak{R}$ for which $\xi = 0$ and $r > 0$; by C_3 the subclass for which $r = 1$; and by C_4 the subclass composed of zero and all the other numbers of $\mathfrak{R}$ for which $\xi > 0$. Evidently all four of these subclasses are closed under the $\times$-operation.

By considering in connection with C and C' a single-valued function φ_{ijk}, we have a system Σ. Moreover, since the distributive law holds for $\mathfrak{R}$, Condition A is satisfied. It will be necessary to verify Condition B for each case, to show that $[\zeta, p] = \varphi_{ijk}([\xi, r], [\eta, s])$ is independent of η if $s = 0$, and then investigate whether each of the three postulates I, II, III is or is not satisfied by φ_{ijk} for all values of its arguments.

Case 111. In addition to the example given in § 2, we may mention the function

$$\varphi_{111}([\xi, r], [\eta, s]) = [\zeta, p]$$

on $C_1 C'$ to C_1 where

$$\zeta = s\xi \cos \eta + s \sin \eta \log r, \quad \log p = s \cos \eta \log r - s\xi \sin \eta.$$

If we choose $[0, e]$ to be the element c_0 of Condition B, we find that we may choose as c' any element $[\eta, s]$ defined by the relations $s \sin \eta = \zeta$, $s \cos \eta = \log p$. Hence c' always exists in C' for every element $[\zeta, p]$ in C_1.

When $s = 0$, $[\zeta, p] = [0, 1]$ is independent of η. It may be shown by a direct calculation that φ_{111} satisfies Postulates I, II and III. In fact, this function corresponds to the power function of complex variable theory. The latter however has a denumerable infinity of values, whereas the function φ_{111} on C_1C' to C_1 is single-valued.

Case 011. The function φ_{011} on C_2C' to C_2 is defined by the identity

$$\varphi_{011}([0, r], [\eta, s]) \equiv [\zeta, p] \equiv [0, r^s].$$

If we take $c_0 = [0, e]$, then $c' = [\eta, \log p]$ where η is arbitrary.

Case 101. The function φ_{101} on C_1C' to C_1 is defined by the identity

$$\varphi_{101}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [s\xi \cos\eta + s \sin\eta \log r, r^{s \cos \eta}].$$

If $c_0 = [0, e]$, then c' is any element $[\eta, s]$ such that

$$s \sin \eta = \zeta, \qquad s \cos \eta = \log p.$$

Case 110 is empty in view of the theorem of § 3.

Case 001. The function φ_{001} on C_3C' to C_3 is defined as

$$\varphi_{001}([\xi, 1], [\eta, s]) \equiv [\zeta, 1] \equiv [\xi s\eta, 1].$$

If $c_0 = [1, 1]$, then c' is any element $[\eta, s]$ such that $\eta s = \zeta$.

Case 010. The function φ_{010} on C_4C' to C_4 is defined as

$$\varphi_{010}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [\xi e^\eta, rs].$$

If $c_0 = [e, 1]$, then $c' = [\log \zeta - 1, p]$.

Case 100. The function φ_{100} on C_1C' to C_1 is defined as

$$\varphi_{100}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [s \sin \eta, r^{\xi s \cos \eta}].$$

If $c_0 = [1, e]$, then c' is any element $[\eta, s]$ defined by

$$s \sin \eta = \zeta, \qquad s \cos \eta = \log p.$$

Case 000. For the function φ_{000} on $\mathfrak{R}C$ to $\mathfrak{R}$, we may take

$$\varphi_{000}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [\eta, s].$$

Let c_0 be any convenient element of $\mathfrak{R}$. Then $c' = [\zeta, p]$.

Thus the problem of the complete independence of these postulates is completely solved, subject to the mild restrictions of Conditions A and B. It is found that I *and* II *jointly imply* III, *and no other dependence relations exist.*

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