follows, by a theorem due to E. W. Chittenden, that* S converges relatively uniformly on the sum of all the point sets of this collection. But this sum is $E - E_0$.

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THE THEORY OF CLOSURE OF TCHEBYCHEFF POLYNOMIALS FOR AN INFINITE INTERVAL†

BY J. A. SHOHAT (J. CHOKHATE)

1. *The Theorem of Closure.* Suppose we have a function $p(x)$, not negative in a given interval (a, b), for which all the integrals

$$\int_a^b p(x)x^n dx, \qquad (n = 0, 1, 2, \cdots)$$

exist. It is well known that we can form a normal and orthogonal system of polynomials

$$\varphi_n(x) = a_n x^n + \cdots, \quad a_n > 0, \quad (n = 0, 1, 2, \cdots),$$

uniquely determined by means of the relations

$$\int_a^b p(x)\varphi_m(x)\varphi_n(x)dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

We call these polynomials *Tchebycheff polynomials* corresponding to the interval (a, b) with the *characteristic function* $p(x)$. The simplest example is given by Legendre polynomials, corresponding to the interval $(-1, +1)$ with $p(x) = 1$.

The most important application of Tchebycheff polynomials is their use in the development of functions into

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* E. W. Chittenden, *Relatively uniform convergence of sequences of functions,* TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 197–201. As Chittenden observes, this is an extension of a theorem given by E. H. Moore on page 87 of his *Introduction to a Form of General Analysis,* loc. cit.

† Presented to the Society, December 29, 1923.

a series of polynomials whose coefficients are easily obtained as in Fourier series. Thus we have for every function, for which the integrals

$$\int_a^b p(x)f(x)x^n dx, \qquad (n=0,1,2,\cdots)$$

exist, the formal development into a series

$$(1) \qquad f(x) \sim \sum_0^\infty {}_n A_n \varphi_n(x), \quad A_n = \int_a^b p(x)f(x)\varphi_n(x)dx.$$

For the case of a *finite interval* (a, b) there always exists the following remarkable equation, which is called the *closure equation*:

$$(2) \qquad \int_a^b p(x)f^2(x)dx = \sum_0^\infty {}_n A_n^2,$$

for every function $f(x)$, for which the integral in the left-hand member exists, even if the development (1) diverges.

The existence of (2) was proved by W. Stekloff.* He showed also the great importance of the "closure equation" in the general theory of Tchebycheff polynomials, in particular for investigating the convergence of the series (1). These results suggest that it might be of interest to investigate the case of an *infinite interval.*

Stekloff† considered only two special cases involving infinite intervals:

(a) polynomials of Laguerre-Tchebycheff
$(a=0,\ b=\infty;\ p(x)=e^{-x}x^{\beta-1},\ \beta>0)$;

(b) polynomials of Laplace-Hermite-Tchebycheff
$(a=-\infty,\ b=+\infty;\ p(x)=e^{-x^2})$.

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* W. Stekloff, *Sur la théorie de fermeture des systèmes des fonctions orthogonales . . .*, MÉMOIRES DE L'ACADÉMIE DES SCIENCES, Petrograd, (7), vol. 30 (1911).

† W. Stekloff, *Théorème de fermeture pour les polynomes de Laplace-Hermite-Tchebycheff*, BULLETIN DE L'ACADÉMIE DES SCIENCES, Petrograd, 1916, pp. 403-16.

——— *Théorème de fermeture pour les polynomes de Tchebycheff-Laguerre,* ibid., 1916, pp. 633-42.

He proved that the closure equation holds in those cases. I have investigated the general case of an *infinite interval* (a, ∞). The results obtained can be formulated in the following theorem.

THEOREM. 1°. *The closure equation holds if, for x sufficiently large,*

$$p(x) < e^{-k|x|^{\lambda}}$$

with $k > 0$ and with $\lambda > \frac{1}{2}$ for $a = 0$, $\lambda > 1$ for $a = -\infty$;
2°. *The closure equation is not true in general, if*

$$p(x) > e^{-k|x|^{\lambda}}$$

with $k > 0$ and with $\lambda < \frac{1}{2}$ for $a = 0$, $\lambda < 1$ for $a = -\infty$.

The assertion (1°) contains the two cases considered by Stekloff. The proof of the first part (1°) of the above statement is based on the paper *Theory of functions deviating the least from zero in an infinite interval*, developed by myself in 1921.* The principal results obtained there are as follows.

Consider two function $f(x)$ and $p(x)$ defined in the infinite interval $(-\infty, +\infty)$ and subject to the following conditions:

(a) $f(x)$ and $p(x)$ are single-valued and finite for every x;

(b) for $x = \pm\infty$ $\lim f(x) = \lim p(x)x^n = 0, \qquad (n = 0, 1, 2, \cdots)$.

I proved that *there exists at least one polynomial $Q_n(x)$ of degree n such that the function*

$$\varphi(x) \equiv f(x) - p(x)Q_n(x)$$

deviates the least from zero in $(-\infty, \infty)$, i. e., $Q_n(x)$ minimizes

$$E_n = \max. |\varphi(x)| \text{ in } (-\infty, \infty) \tag{3}$$

among all polynomials of degree n. Furthermore,

$$\lim_{n \to \infty} E_n = E \geqq 0$$

exists. In general we have $E > 0$. But under certain additional conditions imposed on $f(x)$ and $p(x)$ we have

$$\lim_{n \to \infty} E_n = 0$$

* ANNALS OF THE URAL-UNIVERSITY, No. 1, 1921, Ekaterinburg.

as in the case of a finite interval with $p(x)=1$. The results hold for any infinite interval (a, ∞).

As an application of the results above to Tchebycheff polynomials I gave in the same paper the following theorem.

THEOREM. *The closure equation holds for any orthogonal and normal system of Tchebycheff polynomials corresponding to the interval* (a, ∞) *with the characteristic function*

$$p(x) = e^{-k|x|^{\lambda}}q(x), \qquad (k>0,\ q(x)\geqq 0),$$

if the integral

$$\int_a^{\infty} e^{-(k-\varepsilon)|x|^{\lambda}}q(x)dx$$

exists for a certain positive ε, *with* $\lambda>\frac{1}{2}$ *for* $a=0$, $\lambda>1$ *for* $a=-\infty$.

From this theorem our statement (1°) follows immediately; for, if we set

$$p(x) = e^{-k|x|^{\lambda}}q(x),$$

we get $q(x)<1$ for x sufficiently large, and consequently the integral

$$\int_a^{\infty} e^{-(k-\varepsilon)|x|^{\lambda}}q(x)dx$$

exists for every positive $\varepsilon<k$.

In order to prove the statement (2°) concerning the non-existence of the closure equation I consider first the interval $(0, \infty)$, and give an example of an orthogonal and normal system of Tchebycheff polynomials for which the closure equation is not true.

Let us set

$$p(x) = e^{-x^{\lambda}},$$
$$f(x) = \sin(x^{\lambda}\tan\lambda\pi),$$

with $\lambda<\frac{1}{2}$. Then, by a theorem of Adamoff,*

$$\int_0^{\infty} p(x)f(x)x^n dx = 0, \qquad (n=0,1,2,\ldots),$$

* A. Adamoff, *Proof of a theorem due to Stieltjes*, PROCEEDINGS OF THE MATHEMATICAL SOCIETY AT KAZAN, 1901, (in Russian).

from which we deduce immediately that *all the coefficients in the development* (1) *vanish*:

$$A_n = 0, \qquad (n = 0, 1, 2, \ldots).$$

If we assume the existence of the closure equation in this case we get, from (2),

$$\int_0^\infty p(x) f^2(x) dx = 0,$$

which is evidently impossible.

In the case of the interval $(-\infty, \infty)$ we set

$$p(x) = e^{-x^\beta} \quad \left(\beta = \frac{2s}{2s+1}, \quad s \text{ a positive integer}\right),$$

$$f(x) = \cos\left(x^\beta \tan\frac{\beta\pi}{2}\right);$$

then, from a result of W. Stekloff,*

$$\int_{-\infty}^\infty p(x) f(x) x^n dx = 0, \quad (n = 0, 1, 2 \ldots).$$

The rest of the proof is as before.

The case

$$p(x) = e^{-|x|^\lambda}$$

with $\lambda = \frac{1}{2}$ for $(0, \infty)$, $\lambda = 1$ for $(-\infty, \infty)$ requires a more delicate analysis.

2. *Application to the Theory of Functions Deviating the Least from Zero in an Infinite Interval.* In my paper mentioned above I showed that

$$\lim_{n \to \infty} E_n = E > 0,$$

(see (3)), if the closure equation does not hold for the system of Tchebycheff polynomials with the characteristic function $p(x)$. Thus we conclude

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* W. Stekloff, *Application de la théorie de fermeture à la solution de certaines problèmes des moments,* Mémoires de l'Académie des Sciences, Petrograd, (7), vol. 33 (1915).

In the case of

$$p(x) > e^{-k|x|^{\lambda}}, \qquad (k>0)$$

with $\lambda < \frac{1}{2}$ *for* $(0, \infty)$, $\lambda < 1$ *for* $(-\infty, \infty)$ *we have in general*

$$\lim_{n\to\infty} E_n > 0.$$

3. *Connection with the Theory of Continued Fractions.* In conclusion I want to point out the intimate connection between the theory of closure of Tchebycheff polynomials corresponding to the interval (a, b) with the characteristic function $p(x)$, and the theory of Stieltjes' continued fraction

$$\cfrac{1}{b_1x + \cfrac{1}{b_2x + \cfrac{1}{b_3x + \cdots}}},$$

which arises from the integral

$$\int_a^b \frac{p(y)}{x-y}\,dy, \qquad (a \geqq 0,\ b > a).$$

Indeed, if (a, b) is a finite interval, then the continued fraction for every $p(y)$ converges, and in this case the closure equation always holds. On the other hand, if the interval considered is infinite, say $(0, \infty)$, and if

$$p(y) = e^{-k|y|^{\lambda}}, \qquad (k>0),$$

we know, according to Stieltjes,* that the continued fraction converges for $\lambda \geqq \frac{1}{2}$ and diverges for $\lambda < \frac{1}{2}$. This is in accordance with the statements $\cdot(1^0, 2^0)$ given above.

The closure theory is also closely connected with so called *problem of moments*, due also to Stieltjes.† I hope to show this in another paper.

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* Stieltjes, *Recherches sur les fractions continues*, ANNALES DE LA TOULOUSE, vol. 8 (1894), pp. J. 1–J. 122, vol. 9, pp. A. 1–A. 47.

† Loc. cit., vol. 9, pp. A. 1–A. 47.