EISENHART'S TRANSFORMATIONS OF SURFACES

Transformations of Surfaces. By Luther Pfahler Eisenhart. Published with the cooperation of the National Research Council by The Princeton University Press, 1923. IX + 379 pp.

The results of a round five score of researches in three-dimensional differential geometry, generalized to a great extent to n-space and developed largely anew to form a unified theory governed by a central idea,—this is the offering before us. The researches, with few exceptions, are the product of investigations of the last quarter of a century carried on primarily by Bianchi, Darboux, Demoulin, Eisenhart, Guichard, Jonas, Koenigs, Ribaucour, and Tzitzeica. They deal, some directly and some rather indirectly, with transformations of surfaces of a given kind into surfaces of the same kind.

Of these transformations there are two general types, transformations F, the nature of which we shall describe later, and transformations in which the two surfaces are the focal surfaces of a W congruence. Both transformations appeared first in special forms, the latter in the particular case of pseudospherical surfaces developed by Bianchi (1879) and Bäcklund (1883), the transformations F in special cases discussed by Koenigs (1891), Darboux (1899), and subsequent writers. In fact, it was not until quite recently that the general transformation F was systematically studied, by Jonas, in 1915, and by Eisenhart, in 1917. It is this theory of F, or fundamental, transformations which forms the central and unifying theme of the book.

The scope of the book is broader than the title might first suggest. The material handled is not merely abundant—besides the text there are at the end of each of the ten chapters an average of twenty-five problems serving to a large extent to summarize the results of research,—but it is also highly diversified. More than a third of the book is devoted to congruences of spheres and of circles, rolling surfaces, and surfaces applicable to a quadric, subjects which one would not at first thought relate to the theory of transformations. To unify such a diversity of material in a natural and effective fashion is not simple, and the author is to be congratulated on the masterly way in which he has succeeded. Thereby he has not only given us in many cases new methods of arriving at known results, but has brought home to us in striking fashion the breadth and power of the central theory of F transformations.

The present book is a sequel to the author's *Differential Geometry* in the sense that it refers freely to the elementary treatise for facts and formulas. It is written, too, in much the same style. The development is concise and lucid, and the material in the separate chapters is well ordered. The references to the literature leave nothing to be desired, as soon as one comprehends that the many articles referred to by title only are to be attributed to the author himself.

J. L. Coolidge quotes Darboux as once having told him that in his opinion the essence of geometry consisted in finding in each individual problem the best method for its solution. He might well have added that this is a *sine qua non* in many a problem in differential geometry. For in this field, as much as, if not more than, in any other, a happy choice of method is essential. All the more then is Eisenhart, and those on whose researches he has drawn, to be complimented on the relative simplicity of the analytic formulation of problems. In proceeding now to more detailed discussions we hope to be able, in a few instances at least, to exhibit this analytic simplicity, as well as the elegance of the geometrical results.

When two surfaces in 3-space are in one-to-one point correspondence, there exists in general on each surface a (conjugate) net which corresponds to a (conjugate) net on the other surface. It is by means of these nets and the congruence of lines joining corresponding points on the two surfaces that the transformation is studied. A net is characterized by its point equation, written in the form

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial \log a}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial\theta}{\partial v}. \tag{1}$$

Congruences of lines are restricted to those having two distinct families of developables. A congruence G and a net N are said to be conjugate if the curves of N lie on the developables of G, provided that N is not a focal net of G.

If N and N_1 are two nets in correspondence so that the lines joining corresponding points form a congruence G whose developables contain the nets, three possibilities arise. If N and N_1 are the focal nets of G, they are Laplace transforms of one another. If N is a focal net of G and N_1 is not, N_1 is called a Levy transform of N.

In the general case, when neither N nor N_1 is a focal net of G, G is conjugate to both N and N_1. This is the fundamental transformation, or transformation F.

To obtain a transformation F of a net N, we have but to choose first a congruence G conjugate to N and then a net N_1 conjugate to G; G is determined by a net N' parallel to N in that its direction parameters can be taken equal to the point coordinates of N'; then each net N_1 conjugate to G, and hence in relation F to N, is determined by a solution θ of the point equation (1) of N. The point coordinates of N_1 have the simple form $x_1 = x - (\theta/\theta')x'$, where θ' is a solution, determined by θ, of the point equation of N'. Thus an F transform

N_1 of N involves two independent elements, a congruence G conjugate to N and a solution θ of (1).

Two nets N_1 and N_2, which are F transforms of the same net N by G_1, θ_1 and G_2, θ_2 respectively, are obviously F transforms of one another if G_1 and G_2 are the same congruence. If G_1 and G_2 are distinct, N_1 and N_2 will be in relation F if and only if $\theta_1 = \theta_2$, and then any two of the three nets, N, N_1, N_2, will be F transforms of the third by the same solution of the point equation of the third. The general case, in which G_1 and G_2 are distinct and $\theta_1 \neq \theta_2$, gives rise to the first and most important of the so-called theorems of permutability with which the book is studded. According to this theorem, if N_1 and N_2 are F transforms of N, as described, there exists a net N_{12} which is an F transform of both N_1 and N_2. Corresponding points of the four nets lie in a plane p which envelopes a net and corresponding tangent planes meet in a point P which generates a net.

One more important relationship is needed to complete the account of the general theory. A net N and a congruence G are said to be harmonic if the curves of N correspond to the developables of G and the focal points on a line of G lie on the tangents to the corresponding curves of N. This relationship is closely connected with both the transformations of Levy and the transformations F. In particular, it is found that, if two nets are in relation F, they are harmonic to the same congruence, i. e. their corresponding tangent planes meet in lines generating a congruence harmonic to both; this congruence is known as the harmonic congruence of the transformation F.

If the two non-parallel congruences G_1 and G_2 are harmonic to a net N, the locus of the point of intersection of corresponding lines of G_1 and G_2 describes a net, conjugate to G_1 and G_2 and known as a derived net of N. Conversely, if G_1 and G_2 are conjugate to a net N, the planes determined by corresponding lines of G_1 and G_2 envelope a net, harmonic to G_1 and G_2 and known as a derivant net of N. The net generated by the point P above mentioned is a derived net, and that enveloped by the plane p a derivant net, of each of the four nets entering into the theorem of permutability.

The general theory that we have outlined admits immediate extension to n-space, and it is in this form that it appears, in the first two chapters of the book. Chapter III deals with Laplace sequences in n-space; the analytic conditions that a sequence be periodic are established and hence it is shown that a periodic sequence of order p lies in a space of $p-1$ dimensions. In this and the two subsequent chapters homogeneous point and tangential coordinates are introduced and applied to suit the needs of the material. The developments of the latter chapters, IV and V, are for 3-space and have primarily to do with transformations of special type.

Of particular transformations F we mention two which are essentially duals of one another, namely transformations K and transformations Ω. A transformation K (Koenigs) is characterized by the property that a pair of corresponding points of the two nets separate harmonically the corresponding focal points of the conjugate congruence. In the case of a transformation Ω (Eisenhart), a pair of corresponding tangent planes to the two nets separate harmonically the corresponding focal planes of the harmonic congruence. In the first case the nets have equal point invariants and in the second, equal plane invariants.

An interesting correspondence is developed between transformations Ω of a net N into a net N_1 and transformations of Bianchi in which a surface Σ and its transform Σ_1 are the focal surfaces of a W congruence. The correspondence is characterized by the property that Σ and Σ_1 have the same spherical representations of their asymptotic lines as the curves of N and N_1 respectively. By means of it the theorem of permutability for transformations Ω is readily translated into a theorem of permutability for the transformations of Bianchi.

As further subjects treated in Chapters IV and V we should like to mention the transformations Ω of permanent nets (nets admitting an infinity of applicable nets), the theory of R nets (Demoulin and Tzitzeica), and that of ray congruences and ray curves (Wilczynski).

The interesting and important question of transformations of orthogonal nets (O nets) is considered exhaustively, first in n-space in Chapter VI and then in 3-space in Chapter VII. To discuss transformations F of O nets, it is necessary to know both the nature of the congruences conjugate to O nets and the nature of the nets conjugate to these congruences. It is shown (Guichard) that the latter nets are either O nets in the same space as the given one or projections in this space of O nets in higher spaces, and that the conjugate congruences are related to congruences of isotropic lines in a correspondingly simple manner. To Guichard is also due the introduction of the powerful analytical tool which renders this difficult subject tractable, namely, a suitably chosen moving polyhedral attached to the given O net.

Of the transformations F of O nets into O nets the most important are the transformations R, named for Ribaucour. These transformations have two fundamental aspects, which we shall describe for 3-space. In the first place, two O nets in relation R consist of the principal curves on the two sheets of the envelope of a congruence of spheres; the centers of the spheres form a net, called the central net, which is an F transform of each of the O nets and is itself the projection of an O net in 4-space. Again, the O nets of a one-parameter family consisting of a given O net, N, and its ∞^1 R transforms N_1 by the same θ possess a common harmonic congruence and admit an orthogonal congruence of circles (a cyclic system) whose axes are the lines of this congruence; a point of N and the corresponding points of the nets N_1 all lie on one of the circles.

There are two important subclasses of transformations R, the transformations D_m (Darboux and Bianchi) and the transformations E_m (Eisenhart and Bianchi). The first are the transformations R of isothermic O nets into isothermic O nets and are of type K; the second are the transformations R of O nets with isothermal spherical representation into O nets of the same kind and are of type Ω. In both cases the subscript m refers to a parameter in the transformation.

Chapter VIII deals with congruences of spheres and congruences of circles (in 3-space). A striking feature of this chapter is a correspondence established between congruences of spheres in 3-space and congruences of lines in 5-space, whereby the principal curves on the envelope of the spheres correspond to the developables of the congruence of lines and the pentaspherical coordinates of the spheres are the direction parameters of the lines. A congruence of circles of restricted type appears as consisting of the circles of intersection of the corresponding spheres of the two congruences which correspond to the congruences of tangent lines to a net in 5-space. In fact, this is the author's definition and from it he readily develops the characteristic property that a circle of the congruence is intersected by each of two infinitely near circles in two points. According to this definition, a congruence of circles and a net in 5-space correspond. Hence relationships between congruences of spheres and circles can be interpreted in 5-space as relationships between rectilinear congruences and nets, and vice versa. It was doubtless with this in mind that the author approached the relationships between congruences of spheres and circles previously developed by Guichard and justified the characterization of these as harmonic, conjugate, and orthogonal, on the basis that they correspond to like-named relationships in 5-space. Similarly, the transformations F of congruences of circles which are developed correspond to the transformations F of nets in 5-space. In particular, since to a cyclic system of circles corresponds an O net, the problem of F transformations of cyclic systems is equivalent to that of F transformations of O nets.

If S and $\bar{S}$ are two surfaces applicable to one another, and if S is held fast while $\bar{S}$ assumes all the (∞^2) positions in each of which a point of it is made to coincide with the corresponding point of S so that the corresponding directions at the two points also coincide, $\bar{S}$ is said to roll on S. In Chapter IX the author develops the theory of rolling surfaces, in the general case in which there are actually corresponding nets on S and $\bar{S}$, by applying the general theory of transformations F to nets admitting applicable nets. He thus obtains the theorems of Darboux and Guichard relative to elements invariably fixed to $\bar{S}$, as $\bar{S}$ rolls on S. As an example of the beautiful results deducible in this manner, we quote one theorem: When $\bar{S}$ rolls on S, the isotropic lines through a point invariably fixed to $\bar{S}$ meet the plane of contact of $\bar{S}$ with S in a circle which generates a cyclic system and whose points generate the O nets orthogonal to this system. The latter part of the chapter is devoted to the theory, due primarily to Bianchi, of transformations R deformable in the sense that their central nets admit applicable nets which can serve as central nets of new transformations R; in particular, deformable transformations D_m and E_m are discussed.

The problem of the deformation of a quadric, considered in Chapter X, is approached through the study of permanent nets on the quadric and the corresponding permanent nets on the deforms of the quadric. Two transformations of these nets are then investigated, a transformation F_k, of Guichard and Eisenhart, and a transformation B_k of Bianchi. The first of these is simply a transformation F of a permanent net on a quadric, Q, into a second permanent net on Q, or a resulting transformation F of a net applicable to Q into a similar net; for each value of the constant k there exist ∞^2 transformations F_k of a net $\bar{N}$ applicable to Q into ∞^2 nets $\bar{N}_1$ applicable to Q, and for each of certain special values of k the nets $\bar{N}_1$ can be arranged in one-parameter families possessing elegant properties.

Two nets in relation B_k are the focal nets of a W congruence; in other words, a transformation B_k is not of type F, but of the second type discussed at the beginning of the review. Moreover, as originally developed by Bianchi, the transformations B_k have apparently no connection with transformations F_k. Our author may, then, take justifiable pride in bringing the two into coordination. To this end, he introduces a new definition of a transformation B_k. Thereby, a B_k transform of a net $\bar{N}$ applicable to a net N on Q appears as the derived net, $\hat{N}$, of $\bar{N}$ by two suitably chosen solutions, θ_1 and θ_2, of the point equation of $\bar{N}$; $\hat{N}$ is, of course, applicable to Q, and the net on Q of which it is the deform is related in a simple manner to the derived net of N by θ_1 and θ_2, a net lying on a quadric confocal to Q. The connecting link between the transformations B_k, thus defined, and the transformations F_k consists in the fact that each of the functions, θ_1, θ_2 defines also an F_k transform of $\bar{N}$. This connection is not merely of interest in itself, but serves also to establish a theorem of permutability of transformations of the two types.

The major elements of the book are beyond criticism. The author has, however, a tendency on occasion to scorn minor conditions and subsidiary cases. For example, the definition of a net is so formulated that it includes systems of curves not generally regarded as nets, namely systems on a developable surface one of whose families consists of the asymptotic lines; the author doubtless intended to exclude systems of this type, but nowhere is there a statement to this effect.

In the definition of a net p, O (the projection in n-space of an O net in $(n+p-1)$-space), it appears essential that the $p-1$ complementary functions should be themselves linearly independent as well as linearly independent of the point coordinates of the net. Even then, there would exist nets, each of which belonged, under the definition, to every one of the categories p, O, $p \geqq 2$; the net of generators of a paraboloid, considered as a surface of translation, is of this type. It is of course possible to leave the definition so that a particular net admits various characterizations p, O, but this is not in keeping with the spirit of the subsequent investigations, in which always the minimum value of p is sought. Similar remarks hold for the congruences p, I.

On page 10 we find the theorem: If two non-planar nets correspond and the tangents to the parametric curves in one family are parallel, the nets are parallel. This theorem is true only in general and the exceptions to it are of some importance. For example, in the map of the two translation surfaces, $x = U(u) + V(v)$, $\bar{x} = U(u) + \bar{V}(v)$, the generators correspond and corresponding tangents to the generators $v =$ const. are parallel; the map, however, is not in general a parallel map. The slip in the proof comes from overlooking the fact that a certain equation may be illusory. Fortunately, the theorem does not figure in subsequent developments.

On page 24 it is stated that two solutions of the equation (1) are functionally dependent if and only if one is a linear function of the other with constant coefficients; this is obviously untrue in case either of the coefficients on the right hand side of (1) is zero. On page 198 the possibility of the vanishing of one or the other of two functions (q and r) is not considered, so that the results there obtained appear to be true only in general.

The Hamburg press, from which the work comes, has done a creditable piece of book manufacturing, except for the bright (now dirty) yellow of the binding. The proof reading of the text itself might have been done more thoroughly, but there are next to no errors in the formulas, so far as the reviewer has been able to ascertain by checking a reasonable selection from the thousand and more which the book contains.

As the first of what it is to be hoped will be many mathematical books published with the cooperation of the National Research Council, this work sets a worthy standard. The need of unified presentations of the results of research in special fields is becoming more and more pressing and one who has so successfully met this need in his own field as the present author may look back happily on his labors with the comforting realization that he has contributed greatly to his science.

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