THE APRIL MEETING IN CHICAGO

The twenty-first regular Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 18 and 19, 1924. Nearly one hundred persons attended this meeting; among them were the following seventy-six members of the Society:

R.W. Babcock, W.S. Beckwith, H.A. Bender, Bliss, Blumberg, Bradshaw, O.E. Brown, Bunyan, C.C. Camp, Carlson, Carmichael, Crathorne, Curtiss, Denton, Dickson, Dostal, Dowling, Dresden, Emch, Escott, W.B. Ford, Gibbens, Gossard, Gouwens, L.M. Graves, W.A. Hamilton, E.R. Hedrick, Hildebrandt, Ingraham, Dunham Jackson, Kennedy, Kinney, Lane, Lefschetz, Logsdon, Luby, Lunn, McGavock, J.V. McKelvey, W.D. MacMillan, T.E. Mason, B.I. Miller, Miser, E.H. Moore, E.J. Moulton, F.R. Moulton, Nowlan, C.I. Palmer, Pepper, Pettit, Plapp, R.G.D. Richardson, Rider, H.L. Rietz, Roever, Roth, Schottenfels, Sellew, Sharpe, Shohat, W.G. Simon, Skinner, Slaught, Snedecor, Virgil Snyder, Stouffer, J.H. Taylor, Teach, T.Y. Thomas, J.S. Turner, Wainwright, Wait, K.P. Williams, F.E. Wood, J.W.A. Young, Ziwet.

At the meeting of the Council, the following thirty-seven persons were elected to membership in the Society:

Professor Eli Allison, Brenau College;
Professor Normal Herbert Anning, University of Michigan;
Professor Ernest Aubrey Bailey, LaGrange College;
Mr. Ben Charles Bellamy, Civil Engineer, Laramie, Wyo.;
Mr. Jacob Alfred Benner, Lafayette College;
Mr. Samuel Fletcher Bibb, University of North Dakota;
Dr. Laura Brant, Instructor in Physics, Vassar College;
Mr. Richard Lucius Cary, Baltimore, Md.;
Mr. Luther Jonathan Deck, Muhlenberg College;
Professor Aloysius F. Frumveller, Marquette University;
Professor Arthur Eugene Gault, Bradley Polytechnic Institute, Peoria, Ill.;
Dr. William Albert Hamilton, University of Wisconsin;
Professor Louis Alan Hazeltine, Stevens Institute of Technology;
Professor Guy Hildebrand Hunt, Southern Branch, University of California;
Professor Byron Ingold, Culver-Stockton College;
Miss Rosa Lea Jackson, Instructor, Northwestern University;
Professor Martha Myrtle Knepper, State Teachers College, Cape Girardeau, Mo.;
Mr. Claiborne Green Latimer, Tulane University;
Professor Otto B. Loewen, Ottawa University, Ottawa, Kansas;
Mr. William Hayward McEwen, Assistant, University of Minnesota;
Mr. Charles Edward Manierre, New York City;
Mr. Gaylor Maish Merriman, University of Cincinnati;

Miss Inez Morris, Kansas State Teachers College, Emporia;
Professor Albert George Rau, Moravian College;
Mr. Pardon Sheldon Remington, The Principia, St. Louis;
Professor David Henry Richert, Bethel College;
Dr. Daniel Stucker Robbins, State College, New Mexico;
Mr. Edwin Lawrence Rose, California Institute of Technology;
Professor Bernice Sanders, Wilberforce University;
Professor Thomas Francis Smith, Little Rock College;
Professor Mary Goode Stallworth, Alabama College;
Mr. Clarence George Tobias Stipe, University of South Dakota;
Professor Mildred Browning Stone, State Normal School, Salem, Mass.;
Professor Ralph Sylvester Underwood, Alabama Polytechnic Institute, Auburn, Ala.;
Mr. William Edwin Van de Walle, Harvard University;
Miss Frances Christine Wenrich, Kansas City Junior College;
Mr. George Andrew Yanosik, New York University;
Mr. Louis Yurow, Washington, D.C.

Thirty-seven applications for membership were received.

The Council accepted the invitation of Cornell University to hold the summer meeting and colloquium in Ithaca in 1925 and expressed to the members of the Mathematical Department of Cornell University its appreciation of the invitation. The speakers and subjects for the colloquium will be Professor L. P. Eisenhart, *The differential geometry of general surfaces*, and Professor Dunham Jackson, *The theory of approximation.*

The Assistant Secretary reported the following appointments made by President Veblen since the last meeting of the Society: as delegate to the celebration of the fiftieth anniversary of the founding of the Société Mathématique de France at the Sorbonne on May 22–24, 1924, Professor E. B. Van Vleck; to represent the Society at the semi-centennial of the founding of Purdue University on May 1–3, Professor William Marshall; Committee on Arrangements for the summer meeting and colloquium of 1925: Professors J.H.Tanner, W.B.Ford, W.A.Hurwitz, R.G.D.Richardson, and H. L. Rietz; Committee to nominate trustees, officers and members of the Council to be elected in December, 1924, Professors G. A. Bliss (Chairman), Arnold Dresden, and C. L. E. Moore.

The Council voted to recommend to the Society for adoption at its meeting on May 3 amendments to the By-Laws, to eliminate the ephemeral matter relating to the transfer of business from the unincorporated Society to the incorporated body; to make possible the transaction by mail of some of the Council's business; and to create a sustaining membership.

The Council adopted resolutions of thanks to Professor O. D. Kellogg for valuable assistance rendered the Endowment Committee, to the Carnegie Corporation and to the firm of Allyn and Bacon for their generous contributions to the endowment fund. The Committee on Endowment was authorized to enter into agreements with industrial firms for a service of information and to proceed with the collecting of data necessary to maintain such service.

Those who were present at the meeting on Friday afternoon united in sending a cablegram extending to Professor Felix Klein their greetings and congratulations on the occasion of his seventy-fifth birthday.

On Friday evening a dinner was held at the Quadrangle Club at which sixty-eight persons were present. The need for continued support of the work of the Endowment Committee was emphasized. It was reported that over $23,000 had been contributed by 362 members of the Society. While this total amount was considered very gratifying, it was thought highly desirable that a much larger percentage of the membership take part in the subscription.

The scientific sessions of Friday and Saturday mornings were presided over by Vice-President Hildebrandt. He was relieved by Professor D. R. Curtiss on Friday afternoon; at this session Professor H. L. Rietz presented the symposium lecture on *Certain topics in the mathematical theory of statistics*; this paper appears in full in the present number of this BULLETIN.

The papers read at this meeting are listed below. Miss Kearney, Mr. Jensen, and Mr. Mickelson were introduced to the Society by Professor Dunham Jackson, and Mr. Reinsch by Professor Carmichael. The papers of Baker, Davis, Gehman, Kearney, Mickelson, Steimley, and Williams (first paper) were read by title.

1. Professor E. P. Lane: *The canonical eight-point quadric of a space curve.*

An eight-point quadric of a space curve, at a point P of the curve, is the limiting position of a quadric surface through P and seven neighboring points of the curve as these points approach P along the curve. There is a single infinity of eight-point quadrics at each point P of the curve. Among these are two of special interest. One is the osculating, or nine-point, quadric. The other, which the author calls the *canonical* eight-point quadric, is characterized by being tangent at P to the osculating plane of the curve at P. This quadric has an equation of simple form and is intimately connected with the curve. It may be used in defining a local coordinate system, in some respects analogous to the moving trihedron employed in the metric theory of space curves.

2. Professor E. P. Lane: *A characterization of surfaces of translation.*

If there exists on a surface a conjugate net with indeterminate ray curves, then the ray points of every surface point, with respect to this conjugate net, lie in a fixed plane called the ray plane of the net. The author characterizes surfaces of translation as the only surfaces that have the property that on each of them there exists a conjugate net with indeterminate ray curves and with ray plane at infinity.

3. Professor Arnold Emch: *Some problems of closure connected with the Geiser transformation.*

As is well known, the Geiser transformation in a plane is an involutory Cremona transformation defined by a special net of cubics through 7 independent points. Two cubics of the net intersect in a pair of corresponding points (PP') of the transformation. The seven points A_i are the base points of the transformation. To each of these corresponds every point of the rational cubic with this base-point as a node and passing through the six remaining base-points. The jacobian of the net is the so-called Aronhold curve, a definite point-wise invariant sextic with nodes at the base-points. This paper considers the classes of invariant curves as loci of pairs of corresponding points, and certain class-curves from which they are generated. The mapping of the Geiser transformation upon a general cubic surface leads to interesting problems of closure of sets of certain invariant curves.

4. Professor F. R. Sharpe: *The Jonquière space transformation of Godeaux.*

In a recent memoir (Royal Belgian Academy 1921) Godeaux has considered a type of birational transformation in which to any line of a rational congruence and a plane of a pencil in one space corresponds a similar line and plane in a second space. The author shows that the type is included in a type given by Montesano thirty-three years earlier and that a slightly more general form exists. It is also shown that certain cases which Godeaux concludes to be impossible are possible without restriction.

5. Dr. Gladys Gibbens: *A study of the relations between the focal surfaces of two congruences obtained from certain functions of a complex variable.*

This paper develops two rectilinear congruences by means of the Riemann-spherical representation of Wilczynski. They are obtained from an arbitrary function of a complex variable and its osculating linear fractional function. The focal sheets of the latter congruence are always quadrics. The author finds that each of these two quadrics not only osculates the corresponding focal surface of the original congruence, but is the osculating quadric of this surface. The point of osculation on each surface is the point corresponding to the origin, where the original linear fractional function osculates the arbitrary complex function.

6. Professor Virgil Snyder: *Non-monoidal involutorial transformations which leave each monoid of a web invariant.*

The author constructs a web of monoids of order n having a curve and a number of simple basis points in common, and having the property that any three surfaces of the web intersect in two variable points. This pair of conjugate points defines an involution, the properties of which are determined. In addition to the monoidal involutions with 7 basis points, there are six non-monoidal types for every value of $n \geqq 3$. The entire system can be expressed completely in terms of a single set of formulas. A partial list of the possible types for $n=3$ was already known. The other results are new.

7. Mr. J. A. Nyswander: *A direct method of obtaining the solutions of systems of linear differential equations having constant coefficients.*

The author gives a direct and comparatively simple method for obtaining the solutions of any linear differential system of the form $x_i' = a_{i1}x_1 + \cdots + a_{in}x_n$, $(i=1, \ldots, n)$. When the characteristic equation $D(\lambda) = 0$ has distinct roots, there is no difficulty. When $\lambda = \lambda_k$ is a multiple root of $D(\lambda) = 0$, all solutions of the form

$$x_i = \left(A_i^{(1)}t^{\omega} + \cdots + A_i^{(\omega+1)}\right) e^{\lambda_k t}$$

are obtained directly from $\omega + 1$ sets of linear equations in the $A_i^{(j)}$ which are sufficient conditions for the existence of such a solution. The characteristic determinant $D(\lambda)$ affords a set of n identities in λ which, when differentiated with respect to λ, yield (for $\lambda = \lambda_k$) exactly $\omega + 1$ sets of relations involving minors of $D(\lambda_k)$ and their λ-derivatives, whose coefficients are identical with the coefficients of the $A_i^{(j)}$ in the foregoing sets of linear equations; thus the desired values of the $A_i^{(j)}$ are given directly by these sets of relations. All solutions not of the form $x_i = A_i e^{\lambda_j t}$ are of the above type. A general form for the solutions of this type is set up, from which the n solutions of the system can be written at once in terms of minors of the determinant $D(\lambda_k)$ and of their λ-derivatives.

8. Dr. H. A. Bender: *Prime power groups containing only one invariant subgroup of every index which exceeds this prime number.*

If a group G of order p^m, p being an odd prime, is to contain but one invariant subgroup of every index greater than p, then each of these invariant subgroups must be non-cyclic whenever G is non-cyclic. For $p = 2$ the subgroup of order p^{m-1} must be cyclic. In either case the invariant subgroup of index p^2 must be the commutator subgroup of G. Furthermore, G contains a subgroup of index p which includes all of its operators whose orders exceed p^2, and every operator not in this subgroup transforms each operator of this subgroup into itself multiplied by an operator in the preceding major co-set.

It is shown that a necessary and sufficient condition that this group of order p^m contains but one invariant subgroup of every index greater than p, is that its $(m-2)$th commutator subgroup is of order p. The index of the largest invariant abelian subgroup can not exceed $p^{(p+1)/2}$, and the pth power of every operator of G must be in the central of G_{m-1}.

An interesting system of such groups is the Sylow subgroup of order p^{p+1} contained in the symmetric group of degree p^2.

9. Mr. L. M. Graves: *The derivative as independent function in the calculus of variations.*

The integrals considered are of the form

$$I = \int_{x_1}^{x_2} f\left(x, y_1 + \int_{x_1}^{x} z\,dx, z\right)$$

where z is a function $z(x)$ which is bounded and integrable in the sense of Lebesgue on the interval $x_1 \leqq x \leqq x_2$ and satisfies the additional condition $\int_{x_1}^{x_2} z\,dx = y_2 - y_1$. Sets of necessary and of sufficient conditions are obtained for a minimum of the integral I relative to the class of all such functions.

10. Mr. J. Shohat (Jacques Chokhate): *On the development of a continuous function in a series according to Tchebycheff's polynomials.*

Let $f(x)$ be a continuous function on the *finite* interval (a, b), and $\varphi_n(x)$ a normal orthogonal system of Tchebycheff polynomials on (a, b) with the *characteristic function* $p(x) \geqq 0$ in (a, b). The author investigates the convergence of the development $\sum^{\infty} A_i \varphi_i(x)$, where $A_i = \int_b^a p(x) f(x) \varphi_i(x)\,dx$. Writing $f(x) = \sum^{n} A_i \varphi_i(x) + r_n(x)$, and using a method given by D. Jackson in the TRANSACTIONS for 1912, he is able to evaluate $r^n(x)$, if certain assumptions are made concerning the function $\varphi_i(x)$, as e.g. $|\varphi_n(x)| < \tau n^{\sigma} (n = 1, 2, \ldots)$. The method is illustrated by two examples, one of which generalizes results obtained in Jackson's paper.

11. Mr. J. Shohat: *Note on the asymptotic expression of certain definite integrals.*

In his second paper, the author considers the integrals $\int_0^1 f(x) x^n dx$ and $\int_0^1 f(x)[1-(x-\xi)^2]^n\,dx$ under various hypotheses concerning the character of $f(x)$. Formulas are developed which enable him to find asymptotic expressions for a large number of definite integrals, and also an important inequality for $\sum_{i=0}^{n}\varphi_i^2(\xi)$, where $\varphi_i(x)$ are a normal orthogonal system of Tchebycheff polynomials on (0, 1) with the characteristic function $f(x)$ vanishing for $x = \xi$. This note was suggested by an article of P. Funk, *Beiträge zur Theorie der Kugelfunktionen,* MATHEMATISCHE ANNALEN, vol. 77 (1916).

12. Miss Elizabeth Carlson: *On the convergence of certain methods of closest approximation.*

The first part of this paper contains a discussion of conditions on a function $f(x)$ which are sufficient to insure the uniform convergence of $S_n(x)$ to the value of $f(x)$, where $S_n(x)$ is the Sturm-Liouville sum of order n which gives to the integral $\int_0^\pi |f(x) - S_n(x)|^m\, dx (m \geqq 1)$ its minimum value. In the second part of the paper, the same problem is discussed with the Sturm-Liouville sum replaced by a linear combination of the first n characteristic solutions of a certain third order differential equation with a particular set of boundary conditions. In this case it is necessary first to derive theorems concerning the magnitude of $|S_n'(x)|$. Two theorems are proved: If M is the maximum of $|S_n(x)|$ in $0 \leqq x \leqq \pi$, then throughout this interval, $|S_n'(x)| \leqq n^2 p M$, where p is independent of n and the coefficients in $S_n(x)$. Throughout the interval $\varepsilon \leqq x \leqq \pi - \varepsilon$, $|S_n'(x)| \leqq n q M$, where q is independent of n and the coefficients in $S_n(x)$ but depends upon ε.

13. Mr. C. M. Jensen: *The approximate representation of a function by a Sturm-Liouville interpolating formula.*

The approximate representation of a function by means of a Sturm-Liouville interpolating formula gives rise to problems similar to those which have been solved in connection with the representation by the trigonometric interpolating formula. The situation is complicated by the fact that, for the Sturm-Liouville functions, we have only an *approximate* analogue of the trigonometric formulas of the form

$$\sum_{k=1}^{n} \cos \frac{2\pi i k}{n} \cos \frac{2\pi j k}{n} = 0, \ (i \neq j).$$

The main theorem obtained is that the Sturm-Liouville interpolating formula, of order n, applied to a Sturm-Liouville sum of order n, makes an error of the order $(\log n)/n$, the coefficients being subject to certain conditions. If $f(x)$ satisfies a Lipschitz condition, it can be represented by the Sturm-Liouville interpolating formula of order n with an error of order $(\log n)/n$. If $f(x)$ is merely continuous, its approximate representability by the trigonometric interpolating formula necessitates its representability by the corresponding Sturm-Liouville formula, and, for any continuous function, there exists a Sturm-Liouville interpolating formula which will converge to the right value.

14. Miss Dora E. Kearney: *Some theorems on convex functions.*

A continuous function $f(x)$ is said to be convex in an interval (a, b) if it satisfies the relation

$$f[(1-c)x_1 + cx_2] \leqq (1-c)f(x_1) + cf(x_2)$$

for every c in the interval $0 < c < 1$ and for every pair of values x_1, x_2 in (a, b). One usually assumes this relation for $c = 1/2$ as definition and proves it for an arbitrary c. This paper shows that if there is any c in the interval specified, for which the given relation holds, then it holds for every such c. A corresponding theorem is proved for what may be called properly convex functions, for which the equality sign is excluded. Other elementary theorems on convex functions are presented.

15. Mr. E. L. Mickelson: *Orthogonal polynomials for interpolation.*

This paper is concerned with the construction and tabulation of polynomials orthogonal over a finite set of points equally spaced in an interval. The tables will be directly serviceable in the fitting of parabolic curves by the method of least squares or the equivalent method of moments.

16. Professor P. R. Rider: *The correlation between two variates one of which is normally distributed.*

The paper determines the correlation between two variates x and y $(y = kx^n, k > 0)$, where x is distributed according to the so-called normal law of error. A problem of this type would arise if one wished to find the correlation between the diameters and the weights of a set of homogeneous spheres, either the diameters or the weights being normally distributed. A concrete example might be afforded by the apples on a tree. Rietz has given other practical illustrations in discussing the frequency distribution of the second variate (ANNALS OF MATHEMATICS, (2), vol. 23, pp. 292—300). In the present paper it is shown that the product-moment coefficient, r, for the variates x and y is zero for $n \leqq -1/2$ and for $n = 0$, is negative for $-1/2 < n < 0$ and positive for $n > 0$, and approaches the value zero as n approaches ∞; moreover r is equal to unity for $n = 1$ but is less than unity in absolute value for all other values of n.

17. Professor Dunham Jackson: *Elementary applications of the notion of angle in function space.* Preliminary communication.

The present paper, while making no claim to novelty of substance, calls attention to the fact that the essentials of the Pearson-Yule theory of correlation are identical, in the sense of general analysis, with the elementary geometry of function space, and suggests the importance of a coordinated study of two fields which have been developed to a great extent independently of each other.

18. Professor Dunham Jackson: *A symmetric coefficient of correlation for several variables.*

This paper shows how simple geometric considerations in space of n dimensions, or in function space, as the case may be, lead to the definition of a symmetric coefficient for measuring the degree of resemblance of three or more sets of data. As in other instances of correlation, the extreme values of the coefficient are $+1$ and -1. For the case of three variables, the formula is $(4/9)(r_{12} + r_{13} + r_{23}) - 1/3$, where r_{12}, r_{13}, r_{23} are the ordinary coefficients of correlation of the variables taken two at a time.

19. Professor K. P. Williams: *Notation in tensor analysis.*

In this note the author calls attention to a possible simplification of notation in tensor analysis.

20. Professor K. P. Williams: *Concerning a type of integral equation.*

Andreoli has studied to some extent the integral equation in which the upper limit is $g(x)$. In the present paper the author considers the existence of solutions of this equation and also the relation it bears to another type of functional equation.

21. Professor H. T. Davis: *Derivation of the Fredholm theory from a differential equation of infinite order.*

In this paper it is shown how the Fredholm solution of integral equations can be derived directly from a special differential equation of infinite order. The infinite differential equation corresponding to the Fredholm equation is obtained. By taking n successive derivatives of this equation and disregarding terms of order higher than n, a system of equations is obtained which can be solved by elementary means for the unknown function. In the limiting case the Fredholm solution is easily found.

22. Professor G. A. Bliss: *Algebraic functions and their divisors.*

A number of methods have been devised for the development of the theory of algebraic functions without the use of preliminary birational transformations simplifying the singularities of the function. Among the most interesting of these is the one presented by Hensel and Landsberg in their *Theorie der algebraischen Funktionen,* which is an amplification of methods suggested originally, and apparently independently, by Kroneker and by Dedekind and Weber. The present paper is an effort to make more accessible the solutions of two important problems in the Hensel and Landsberg theory: the construction of the three types of elementary abelian integrals and the proof of the Riemann-Roch theorem. It is believed that the proofs have been considerably simplified, and their various logical steps unraveled from a large amount of most interesting but to these problems relatively irrelevant material with which they are interlaced in the Hensel and Landsberg treatise. Incidentally the author of this paper wishes to call attention to a recent reference by him to the Riemann-Roch theorem (TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), footnote, p. 277), which is now justified by a much more concise presentation.

23. Mr. J. H. Taylor: *A generalization of Levi-Civita's parallelism and the Frenet formulas.*

In the n-dimensional space considered in this paper the arc length is defined by an integral whose integrand is a function $F(x_1, \ldots, x_n; x_1', \ldots, x_n')$ which satisfies the conditions of the so-called regular problem of the calculus of variations. In this space vectors can be normed and orthogonalized with respect to a quadratic manifold which osculates the *indicatrix* $F = 1$ for an arbitrary direction x'. Blaschke has associated with the well known covariant derivative of a Riemann space a differentiation process which he calls the θ-process. The author has generalized this process for the more general space considered by him. By means of this generalization he can associate with each point of a curve a system of n contravariant vectors which can also be replaced by an equivalent unitary orthogonal system of vectors. By means of this latter system the Frenet formulas are obtained and the expressions for the curvatures are deduced. If two movable vectors have their initial points on the same curve and satisfy the system of equations $\theta \xi = 0$ associated with the curve it is shown that the angle between them remains a constant. This is a generalization of a theorem by Levi-Civita.

24. Professor R. P. Baker: *The connection of sheets in mapping.*

This paper exhibits various types of connection of sheets of points in multiple mapping by algebraic equations not occurring in ordinary Riemann surfaces. If

$$\varphi\,(u,v:x,y) = 0, \quad \psi\,(u,v:x,y) = 0$$

are algebraic equations and the real points of the locus in $R_4(u,v,x,y)$ are all finite, the (generalized) Riemann surfaces over the (u,v) and (x,y) plane are two-sided and of the same connectivity. They may have removable singularities of various kinds as adhesions and nodal lines with no real neighborhood. If the locus is infinite and the closure of projective space be used, the surfaces may be one-sided. The connectivity is preserved.

The paper considers the theorem of Hadamard (JOURNAL DE MATHÉMATIQUES, 1898) concerning the connectedness of the spherical representation of a surface whose curvature is negative everywhere except at isolated points where it is zero.

In mapping a surface by parameters the connectivity may be changed. For instance, in the two-parameter representation of a general cubic, the mapping is on an anchor ring. Five lines being lost, the connectivity reduces from 8 to 3. A Cremona transformation further reduces it and maps the surface on a projective plane.

25. Mr. H. M. Gehman: *Necessary and sufficient conditions that every closed and connected subset of a continuous curve be a continuous curve.*

It is shown that a necessary condition that every closed and connected subset of a continuous curve M be a continuous curve, is that M contain no closed and connected subset N, containing a continuum of condensation W, such that $N-W$ is uniformly connected im kleinen. This condition is also sufficient, if every continu of condensation of M is a continuous curve, not of a certain type derived by considering the necessary and sufficient conditions that the removal of an arc from a continuous curve leave the remainder (on one side of the arc) not uniformly connected im kleinen. Examples are given showing the independence of each condition.

A study is also made of the effect of replacing "$N-W$ is uniformly connected im kleinen" by "$N-W$ can be expressed as the sum of a finite number of connected sets each of diameter less than a given ε" (the Sierpinski S-property). A method is obtained for constructing a continuous curve by adding to any closed and connected set a countable infinity of arcs. It is shown that this is impossible if the set of arcs be finite in number.

26. Professors E. R. Hedrick and Louis Ingold: *Conjugate functions in three dimensions.*

This paper is a continuation of previous papers dealing with three-dimensional analogs of facts connected with the ordinary theory of functions of a complex variable.

Conjugate functions in two dimensions are certain related pairs of solutions of Laplace's equation. In three dimensions it is found that the solutions of a certain pair of partial differential equations of the second order are related in sets of three, and these sets are the natural analogs in three dimensions of sets of conjugate functions in the plane. Certain theorems are also generalized.

27. Professor J. S. Turner: *On the representation of a positive integer by the form* ax^2-by^2.

This paper contains a simple extension of a theorem of Tchebychef (JOURNAL DE MATHÉMATIQUES, vol. 16 (1851), pp. 257–282). It is proved that if a positive integer m can be represented by the form ax^2-by^2, where a, b are positive integers, then there exists a representation with

$$0 \leqq x \leqq \sqrt{\frac{(T+1)m}{2a}}, \quad 0 \leqq y \leqq \sqrt{\frac{(T-1)m}{2b}},$$

where $x = T$, $y = U$, are the least positive integral solutions of

$$x^2-Dy^2=1, \; D=ab;$$

and that from two such representations factors of m can be found.

It follows that if D is a determinant for which each genus contains only one cycle of reduced forms, and if m is an odd number prime to D and having the quadratic character of ax^2-by^2, then m is prime if it has a single primitive representation by ax^2-by^2 within the limits specified above, and composite in all other cases.

28. Professor Oswald Veblen and Dr. T. Y. Thomas: *Extensions of relative tensors.*

This paper is an addendum to the authors' paper on the geometry of paths in the TRANSACTIONS OF THIS SOCIETY for October, 1923. It contains formulas for the rth extensions of relative tensors of arbitrary weight. These include the formulas for tensor densities.

29. Professor Dunham Jackson: *Note on arc and angle in function space.*

The interpretation of the coefficient of correlation of two functions as the cosine of an angle, discussed in a previous paper, is further illustrated in terms of a parametric formula for length of arc in function space. If $\varphi(x)$ and $\psi(x)$ are two normalized functions in an interval (a, b), represented in function space by two points P and Q at unit distance from the origin, the length of the circular arc from P to Q, with center at the origin, is found by direct calculation to be arc cos r, where

$$r = \int \varphi(x)\psi(x)\,dx.$$

30. Mr. B. P. Reinsch: *Expansion problems in connection with the hypergeometric differential equation.*

The object of this paper is the development of an expansion theory for hypergeometric functions analogous to the one worked out by Neumann and Gegenbauer for Bessel functions. Expansions in series of

$$x^{k+m}F(a+k+m, b, 1+k+m; x), \quad (m=0,1,2,\ldots),$$

solutions of the hypergeometric differential equation, are found for $x^{k+m}, x^k/t-x$; and (employing the last result) for an analytic function $f(x)$ by means of Cauchy's integral formula. The analogues of the Cauchy-Taylor and the Laurent expansions appear. The theory is extended to the generalized hypergeometric functions of Goursat and Pochhammer. The results are then extended to the expansion of analytic functions of several variables in multiple series of ordinary or generalized hypergeometric functions.

31. Dr. L. L. Steimley: *On the summability and regions of summability of a general class of series of the form* $c_0+\sum c_n g(nx)$.

The author considers the general class of series of the form

$$Y(x) = c_0 + \sum_{n=1}^{\infty} c_n g(nx),$$

where $g(x)$ has the asymptotic character

$$g(x) \sim x^{P(x)} e^{Q(x)}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right)$$

valid for x approaching infinity in some sector V bounded by two rays extending from zero to infinity. $P(x)$ and $Q(x)$ are polynomials in x, a_i are functions of the argument of x, varying in the sector V, c_i are independent of x.

This paper deals with the summability of the series $Y(x)$ by the method of Cesàro. The character of the regions of summability and of uniform summability are determined. The character as to summability of every point on the boundary of the region of summability is determined except for a set of isolated points which may occur under very restricted conditions. Summability numbers are determined. A necessary and sufficient condition for summability of $Y(x)$ is developed.

32. Dr. L. L. Steimley: *On a general class of integrals of the form* $\int_0^\infty \varphi(t)g(tx)\,dt$.

The author considers a general class of integrals of the form $K(x) = \int_0^\infty \varphi(t)g(tx)\,dt$, where $g(x)$ has the asymptotic character described in the previous paper. The function $\varphi(t)$ is finite and single-valued for finite $t \geqq 0$, $|\varphi(t)|$ has an upper bound in every finite interval (0τ) and $\varphi(t)$ is integrable in the sense of Riemann in every such interval. The character of the regions of convergence, of absolute convergence and of uniform convergence are determined. The properties as to analyticity of functions defined by integrals $K(x)$ are treated. The character of the points, as to convergence, on the boundary of the region of convergence is determined, except for an isolated set of points. The character of the isolated points is determined except in very special cases. Convergence numbers in terms of φ and g are determined. A necessary and sufficient condition for the convergence of $K(x)$ is established.

33. Professor Cornelius Gouwens: *Invariants of the linear group, modulo* $\pi = p_1^{\lambda_1} \cdots p_n^{\lambda_n}$.

The transformations congruent, modulo π to a given transformation T form a class $[T]_\pi$. These classes are the elements of a group Γ. This group is formed by composition of the subgroups G_i of classes $[T]_\pi$, $T \equiv 1$ (mod $m_i = \pi / p_i^{\lambda_i}$), $|T| \equiv 1$ (mod $p_i^{\lambda_i}$). A function invariant under Γ is invariant under every G_i. The subgroup G_i is simply isomorphic with the group H_i of classes $[S]_p$ of transformations S whose determinants are congruent to unity modulo $P = p_i^{\lambda_i}$. Every invariant of Γ is a sum of invariants each of which is expressible as a product of m_i by an invariant of the group H_i.

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