GROUP OF A SET OF SIMULTANEOUS ALGEBRAIC EQUATIONS*

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1. *Introduction.* Consider a system of m independent and consistent algebraic equations in m variables

$$(1) \qquad f_k(z_1, \cdots, z_m) = 0, \qquad (k = 1, \cdots, m).$$

Let the roots of this system of equations be $(z_{1a}, \ldots z_{ma})$ $(a = 1, \cdots, n)$. We assume that each z_{pa} is finite. Therefore by a process of elimination we obtain equations

$$(2) \qquad F_p(z_p) = 0, \qquad (p = 1, \cdots, m),$$

each of which is of degree n and involves only one of the unknowns. None of these equations vanishes identically, since equations (1) are independent. We assume that none of equations (2) has a multiple root.

A rational domain R which includes the coefficients of (1) includes the coefficients of (2). It will be shown in this paper that the groups of equations (2) relative to R are identical, except for the symbols which they affect.

2. *Elimination of z_m.* On eliminating z_m from (1) we obtain $m - 1$ equations involving $z_1, \cdots, z_{m-1}$. Now z_{m-1} cannot be absent from all of these equations; for in that case we would have $m - 1$ equations involving fewer than $m - 1$ unknowns, and equations (1) would not be consistent and independent as assumed. Repeating this argument we can find two independent and consistent equations

$$(3) \qquad g_1(z_1, z_2) = 0, \qquad g_2(z_1, z_2) = 0,$$

involving the unknowns z_1, z_2 (or any other pair of the unknowns). This elimination can always be carried out in such a way that if (z_{1a}, z_{2a}) is a root of equations (3), then there is a root $(z_{1a}, z_{2a}, \cdots, z_{ma})$ of equations (1).

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3. *Relations of Rationality.* Let R' be the domain obtained by adjoining z_{2a} to R. By equations (3), $g_1(z_{1a}, z_{2a})$ and $g_2(z_{1a}, z_{2a})$ are rational functions of a root of $F_1(z_1) = 0$, with coefficients in R' which equal a number (zero) in R'; they are therefore unaltered in value by the substitutions of the group H of $F_1(z_1) = 0$ relative to R'. Suppose that a substitution of H changes z_{1a} to z_{1b}. Applying this substitution to $g_1(z_{1a}, z_{2a})$ and $g_2(z_{1a}, z_{2a})$ we obtain $g_1(z_{1b}, z_{2a}) = 0$ and $g_2(z_{1b}, z_{2a}) = 0$. By § 2, there are two roots $(z_{1a}, z_{2a}, \cdots, z_{ma})$ and $(z_{1b}, z_{2a}, \cdots, z_{mb})$ of equations (1) with the same value for z_2. Therefore $F_2(z_2) = 0$ has a multiple root, contrary to assumption. It follows that H leaves z_{1a} fixed. Hence z_{1a} is in R' and is a rational function of z_{2a} with coefficients in R.

THEOREM 1. *If $(z_{1a}, z_{2a}, \cdots, z_{ma})$ is a root of equations* (1), *z_{pa} equals a rational function of z_{qa} with coefficients in R $(p, q = 1, 2, \cdots, m)$**.

4. *Group Relations.* Let us regard G_p, the group of $F_p(z_p) = 0$ relative to R, as a substitution group not on the roots $z_{p1}, z_{p2}, \cdots, z_{pn}$, but on their second subscripts. Thus (12) simultaneously interchanges z_{11} and z_{12}, z_{21} and z_{22} etc. With this understanding we state our main result.

THEOREM 2. *The groups of equations* (2) *relative to R are identical.*

We shall prove G_1 identical with G_2. By Theorem 1, $z_{pa} = \varphi_{pa}(z_{1a})$ where φ_{pa} is a rational function with coefficients in R. Hence by equations (1)

$$(4)\quad f_k[z_{1a}, \varphi_{2a}(z_{1a}), \cdots, \varphi_{ma}(z_{1a})] = 0, \quad \begin{pmatrix} k = 1, 2, \cdots, m \\ a = 1, 2, \cdots, n \end{pmatrix}$$

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* The following is a proof which might be offered of this theorem: Eliminate every power of z_1 from equations (3) except the first; then z_1 is expressed as a rational function of z_2 with coefficients in R. The objection to this proof is that in the process of elimination the first power of z_1 may be incidentally eliminated. It is easy to make up examples where this actually occurs. I believe the proof given above is free from objection.

Now suppose a substitution of G_1 replaces z_{1a} by z_{1b}. Then equations (4) become

$$(5) \qquad f_k[z_{1b}, \varphi_{2a}(z_{1b}), \cdots, \varphi_{ma}(z_{1b})] = 0.$$

Therefore $[z_{1b}, \varphi_{2a}(z_{1b}), \cdots, \varphi_{ma}(z_{1b})]$ is a root of (1). But $[z_{1b}, \varphi_{2b}(z_{1b}), \cdots, \varphi_{mb}(z_{1b})]$ is a root of (1), and (2) has no multiple root. Hence $\varphi_{pa}(z_{1b}) = \varphi_{pb}(z_{1b})$. We may therefore omit the second subscript of φ_{pa} and write

$$(6) \qquad z_{pa} = \varphi_p(z_{1a}).$$

Now let $\psi(z_{21}, \cdots, z_{2n})$ be a rational function with coefficients in R of the roots of $F_2(z_2) = 0$ which equals a number in R. Then by (6) $\psi[\varphi_2(z_{11}), \cdots, \varphi_2(z_{1n})]$ is a rational function with coefficients in R of the roots of $F_1(z_1) = 0$ which equals a number in R and hence is unaltered by G_1. But on applying a substitution of G to $\psi[\varphi_2(z_{11}), \cdots, \varphi_2(z_{1n})]$ we find that, in virtue of (6), $z_{21}, \cdots, z_{2n}$ undergo a similar substitution. Therefore $\psi(z_{21}, \cdots, z_{2n})$ is unaltered by every substitution of G_1. It follows that G_1 is a subgroup of G_2. Similarly G_2 is a subgroup of G_1. Hence G_1 and G_2 are identical.

5. *A Necessary and Sufficient Condition.* We shall call $G = G_p$ $(p = 1, \cdots, m)$ the group of equations (1) relative to R. Let $\psi(z_{11}, \cdots, z_{1n}; \cdots; z_{m1}, \cdots, z_{mn})$ be unaltered in value by all the substitutions of G. Then so is $\psi[z_{11}, \cdots, z_{1n}; \cdots; \varphi_m(z_{11}), \cdots, \varphi_m(z_{1n})]$. But the latter is a rational function with coefficients in R of the roots of $F_1(z_1) = 0$, which is unaltered by G. Hence it equals a number in R. Conversely, let $\psi(z_{11}, \cdots, z_{1n}; \cdots; z_{m1}, \cdots, z_{mn})$ equal a number in R. Then so does $\psi[z_{11}, \cdots, z_{1n}; \cdots; \varphi_m(z_{11}), \cdots, \varphi_m(z_{1n})]$. Since the latter is unaltered by G, so is the former.

THEOREM 3. *A necessary and sufficient condition that a rational function with coefficients in R, of the numbers satisfying a set of independent and consistent algebraic equations with coefficients in R, be unaltered in value by the group of the equations relative to R, is that it equal a number in R.*

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