# THE EQUATION OF THE EIGHTH DEGREE*

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1. *Introduction.* The progress of mathematics as a whole is occasionally brought to our attention by the appearance of some notable book or memoir in which the resources of the subject are brought from various fields to bear upon a central problem. An early instance of this is the *Ikosaeder*[1] of Klein—the forerunner of the Klein-Fricke series. The first two chapters of this book furnish an introduction to group theory which is as yet unsurpassed. A later example is the book of Hudson on *Kummer's Quartic Surface.*[2] This surface, remarkable in itself, is more remarkable for the breadth of the theories which attach to it. Central problems in these two books are respectively the quintic and the sextic equation. It is true that Hudson assumes the solution of the sextic when needful, but his book furnishes the geometric background for this solution.

Around the sextic equation there cluster the geometric theory of the Weddle and Kummer surfaces and of certain modular three-ways in $S_4$ as well as the analytic theory of the hyperelliptic integrals and functions of genus two, a complex in which the methods of algebra and group theory have constant play. For the most part the allied geometry is within the spaces of our experience, and with the correspondingly small number of variables the methods of ordinary analytic geometry are effective. The analytic function theory is also fairly manageable by direct methods. When, however, we seek to extend this field of ideas to equations of higher even degree complications develop

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rapidly. The surfaces or manifolds involved lie in spaces of high dimension and are defined by many rather than one equation. The analytic theory is correspondingly difficult. Some solid ground there is, particularly in the theta functions, but not enough to form a basis for a closely knit theory.

It is the purpose of this paper to exhibit certain points of view, certain modes of approach to the general problem, and certain analytic methods which can be applied successfully to the octavic and in most cases to equations of even degree, $2p+2$. An indication there will be of a general theory which is fairly coherent and perhaps not more vague than is inevitable with generalizations of this character.

It should be remarked that the solutions here sought for these equations are solutions in terms of hyperelliptic modular functions and not the solutions of Klein in terms of the form problems of linear groups, which perhaps are more desirable but certainly are less accessible.

2. *Two Linear Systems of Irrational Invariants.* (*a*). The given equation defines three sets of points each set on a rational norm-curve in its space. Thus the sextic defines the set $Q_6^1$ of points $q_1, \ldots, q_6$ on a line $S_1$, the set $R_6^2$ of points $r_1, \ldots, r_6$ on a conic $N^2$ in $S_2$, and the set $S_6^3$ of points $s_1, \ldots, s_6$ on a space cubic curve $N^3$ in $S_3$. Similarly the octavic defines sets of eight points, $Q_8^1$, $R_8^3$, $S_8^5$ on respectively the line $S_1$, the space cubic curve $N^3$ in $S_3$, and the rational norm quintic curve $N^5$ in $S_5$. In general the equation of even degree $2p+2$ defines the sets $Q_{2p+2}^1$ on $S_1$, $R_{2p+2}^p$ on $N^p$ in $S_p$, and $S_{2p+2}^{2p-1}$ on $N^{2p-1}$ in $S_{2p-1}$. We shall take all of these norm curves $N^k$ in $S_k$ in the simple parametric form, $x_i = t^i (i=0, \ldots, k)$, and assume that the parameters $t_1, \ldots, t_{2p+2}$ of the points of each set are either the roots of the given equation or are projective to these roots. We observe that for the extreme sets $Q_{2p+2}^1$, $S_{2p+2}^{2p-1}$ the norm curve requirement imposes no conditions on the points since on $k+3$ points in general position in

$S_k$ there is a unique norm curve $N^k$. The intermediate set $R^p_{2p+2}$ is special due to $(p-1)^2$ conditions that its points shall lie on $N^p$. The extreme sets also furnish one of the simplest examples of *associated sets* ([3]Theorem 10).

(*b*). Invariants of a set of points in $S_k$ are formed from the determinants of the coordinates of $k+1$ points selected from the set, which are combined in such a way that each term is of the same degree in each point. This degree is the *degree* of the invariant. Such invariants are *rational* or *irrational* according as they are or are not unaltered under permutation of this points ([4]I § 3). The sets $Q^1_{2p+2}$ and $R^p_{2p+2}$ have linear invariants which are linear combinations with constant coefficients of products of the respective types

$$\text{(A):}\qquad (12)(34)\cdots(2p+1,\,2p+2),\ (ij)=\begin{vmatrix} t_i & 1\\ t_j & 1\end{vmatrix};$$

$$\text{(I)}\qquad \text{(B):}\qquad (12\cdots,p+1)(p+2,\cdots,2p+2),$$

$$(i_1 i_2\ \cdots\ i_{p+1})=\begin{vmatrix} t^p_{i_1} & t^{p-1}_{i_1} & \cdots & 1\\ t^p_{i_2} & t^{p-1}_{i_2} & \cdots & 1\\ \cdot & \cdot & \cdot & \cdot\\ t^p_{i_{p+1}} & t^{p-1}_{i_{p+1}} & \cdots & 1\end{vmatrix}.$$

The various products of either type are linearly related in many ways by the usual determinant identities. The set $S^{2p-1}_{2p+2}$ contributes nothing new to this invariant theory since its determinants are proportional to the complementary determinants of its associated set $Q^1_{2p+2}$ ([4]I (25)).

(*c*). Obviously if the roots $t_i$ of the given equation are known the numerical values of the invariants (A) are easily found. Conversely if the numerical values of the invariants (A) are known the roots $t_i$ of the given equation may be found by rational processes. Thus from (12) (34) (56) ... $(2p+1,\ 2p+2)$ and (14) (32) (56) ... $(2p+1,\ 2p+2)$ we find by division the double ratio $(13\,|\,24)$. If we take $t'_1,\ t_2,\ t'_3$ to be $0,\ 1,\ \infty$ the values of these double ratios determine the remaining roots $t'_4, \ldots, t'_{2p+2}$ of an equation projective to the given equation. The procedure for finding the transformation which converts the known roots $t'$ into the required roots $t$ of the given equation is a known exercise in binary forms. We shall therefore henceforth regard the determination of the values of the invariants (A) as a "solution" of the given equation.

(*d*). The numerical values of the invariants (B) are in immediate relation to hyperelliptic modular functions of genus $p$. The basis notation[5] for these is very convenient. We divide the numbers 1, 2, . . . , $2p+2$ into equivalent complementary sets and name the half periods by an even number of subscripts as in $P_{1,2,\ldots,2k} = P_{2k+1,\ldots,2p+2}$; the even theta functions of first order by a number of subscripts of the form $p+1\pm 4k$; and the odd functions of the first order by a number of subscripts of the form $p+1\pm 2(2k+1)$, $(k=0, 1, \ldots)$. The simplest set of modular functions are the values of the even functions for the zero arguments. When, however, the functions are hyperelliptic all of these values vanish except the ones with $p+1$ subscripts ([6]pp. 456–464). The non-vanishing values are connected with the invariants (B) by the remarkable relations

$$(2) \quad \varepsilon\cdot(1, 2, \ldots, p+1)(p+2, \ldots, 2p+2) = \vartheta^4_{1,\ldots,p+1}((0)),$$

where $\varepsilon$ is a properly adjusted sign (for $p=2$, cf. [4]I, p. 194; for $p=3$, [5]p. 111).

(*e*). If then we find rational expressions for the invariants of type (A) in terms of those of type (B) we have a solution of the given equation in terms of hyperelliptic modular functions. I shall return in § 3 to a discussion of these relations. Meanwhile let us seek for a natural and easily extensible way of introducing the theta functions themselves into the discussion. For this purpose the set of points $S^{2p-1}_{2p+2}$ plays the important role.

3. *The Generalized Weddle p-way and the Hyperelliptic Kummer p-way.* (*a*). Given a general set of $k+3$ points $s_1, s_2, \ldots, s_{k+3}$ in $S_k$ we define, by isolating a pair of points

$s_1, s_2$, a Cremona involution $I_{12}$ of order $k$, projective to the type $x_i' = 1/x_i$ $(i = 0, 1, \ldots, k)$ which interchanges the points $s_1, s_2$ and has $F$-points at the remaining $k+1$ points of the given set. By considering the effect of these involutions upon the directions about the points of the given set we find that they generate a group for which the following generational relations hold:

$$
\begin{aligned}
I_{12}I_{13} &= I_{13}I_{12} = I_{23},\\
I_{12}I_{34} &= I_{13}I_{24} = I_{14}I_{23} \equiv I_{1234},\\
I_{1234}I_{56} &= I_{1235}I_{46} = \cdots \equiv I_{123456},\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{3}
$$

Hence they generate an abelian group of order $2^{k+2}$, $G_2^{k+2}$, all of whose elements are involutory. We observe, however, an essential difference between the odd and even values of $k$. When $k$ is odd and therefore $k+3$ even, say $2p+2$, there is one element of the group, namely

$$I = I_{123\,\ldots,\,2p+2} \tag{4}$$

which is symmetrically related to the given set of $2p+2$ points, $S_{2p+2}^{2p-1}$ (cf. [4]II, p. 369). We define the Weddle $p$-way, $W_p$, in $S_{2p-1}$, to be the locus of fixed points of the involution $I$. That there is such a locus of fixed points and that it can be expressed parametrically by means of hyperelliptic functions we proceed to prove by a process which is new to the literature.

($b$). The two common canonical forms of the hyperelliptic algebraic plane curve of genus $p$ are first the form

$$y^2 = a_0(x - t_1) \ldots (x - t_{2p+2}), \tag{5}$$

commonly used for the development of the hyperelliptic integrals; and secondly the curve $H_p$ of order $p+2$ with a $p$-fold point at $O$ and $2p+2$ contacts $\tau_1, \tau_2, \ldots, \tau_{2p+2}$ of tangents from $O$. To the so-called "superposed points" $x, \pm y$ of (5) there correspond the pairs of points on $H_p$ on a line through $O$. These pairs constitute the unique $g_1^2$ on $H_p$ invariant under all birational transformations. The form (5) depends upon $2p-1$ "moduli" or "birational invariants". On the other hand the curve $H_p$ has $3p-1$ absolute projective constants. Hence the $\infty^{3p-1}$ projectively distinct curves $H_p$ divide into $\infty^{2p-1}$ birationally distinct classes, each class containing $\infty^p$ curves. The $\infty^p$ curves $H_p$ in each class are birationally equivalent and it may be shown that the curves of a given class may all be obtained from a given curve of the class by selecting on this given curve a $p$-ad of points and transforming this $p$-ad into the $p$-fold point of the transform. This may in fact be done by Cremona transformations of the entire plane. However, each selected $p$-ad determines its "superposed" $p$-ad and either one of a pair of superposed $p$-ads leads to the same transformed curve. Thus the two aggregates—(1) the $\infty^p$ projectively distinct but birationally equivalent curves $H_p$; and (2) one of the curves $H_p$ and the $\infty^p$ pairs of superposed $p$-ads on it—are birationally equivalent. In the next section we connect the first aggregate with the $\infty^p$ points on the Weddle $W_p$; and in the following section the second aggregate with the variables of the attached theta functions and thereby obtain the desired parametric expression of $W_p$.

(c). Each of the $\infty^p$ curves $H_p$ has a $p$-fold point $O$ and $(2p+2)$ branch points $\tau_1, \tau_2, \cdots, \tau_{2p+2}$ which make up a set of points, $T^2_{2p+3}$. The associated set lies in $S_{2p-1}$ and by virtue of a property ([3]p. 1) of such associated sets the points in $S_{2p-1}$ which correspond to $\tau_1, \cdots, \tau_{2p+2}$ can be identified with the fixed set $S^{2p-1}_{2p+2}$. Then as $H_p$ varies in the $\infty^p$ system the point $P$ in $S_{2p-1}$ which corresponds to $O$ in the plane runs over a $p$-way in $S_{2p-1}$. In order to identify this $p$-way with the Weddle $W_p$ we observe that $H_p$ is a locus of fixed points of a Jonquière Cremona involution $J_p$ and therefore the set $T^2_{2p+3}$ is congruent to itself under $J_p$. Moreover $J_p$ is a product of Jonquière quadratic involutions $J_0(O, \tau_1, \tau_2)$, $J_0(O, \tau_3, \tau_4), \cdots$ . Hence (cf. [4]II, p. 361) the associated set in $S_{2p-1}$ is congruent to itself under the product of the involutions $I_{12}, I_{34}, \cdots$, i. e. under $I$. Since the

$F$-points of $I$ are all in the set $S_{2p+2}^{2p-1}$ the point $P$ is a fixed point of $I$ and therefore a point of $W_p$. Conversely for each point $P$ on $W_p$ we have an associated set $T_{2p+3}^2$ and a curve $H_p$. The known[7] relations satisfied by the set $T_{2p+3}^2$ lead to geometric conditions on $P$. Thus for $p=2$ the six branch points $\tau$ are on a conic and therefore $P$ is the vertex of a quadric cone on $S_6^3$— the usual definition of $W_2$. When $p=3$ the eight points $\tau$ and $O$ are the base points of a pencil of cubics, whence $P$ and the 8 points of $S_8^5$ are on a pencil of elliptic sextics in $S_5$ ([3]pp. 17–18).

($d$). Consider now a fixed curve $H_p$ and a $p$-ad of points on it which correspond to the points $(x_1)$, ...., $(x_p)$ on (5). On the Riemann surface of (5) select $p$ paths of integration from the fixed branch point $t_1$ to $(x_1)$, ...., $(x_p)$ respectively. If $v$ stands in succession for one of the $p$ normal integrals of the first kind and we set

$$v_{t_1}^{(x_1)}+\cdots+v_{t_1}^{(x_p)}=u, \tag{6}$$

then to within multiples of the periods the $p$-ad defines a value system $u\equiv u_1, \ldots, u_p$ and conversely $u$ defines as a solution of the inversion problem a $p$-ad $(x_1), \ldots, (x_p)$. Superposed $p$-ads give rise to $\pm u$. The particular $p$-ad at the point $O$ is determined by $u=d$. In this way to a point $P$ on $W_p$ we attach parameters $\pm d$. We readily find (the exposition in [6]Chap. 10 is convenient here) that the involutions $I_{12}$, which in the planar field are represented by $J_0(O, \tau_1, \tau_2)$, transform $W_p$ into itself and are given in terms of the parameters by the simple relations

$$d'\equiv d+\pi_{12} \tag{7}$$

where $\pi_{12}$ is a particular half period. We still have to account, on the Weddle $p$-way, for the $2^{2p}$ singular points and the $2^{2p}$ singular tangent spaces of the Kummer $p$-way. It is more convenient to illustrate these by the relatively simple cases of the sextic and octavic for $p=2, 3$ respectively.

($e$). When $p=2$ the six branch points $\tau_1, \ldots, \tau_6$ of $H_2$ with node at $O$ are on a conic and the discriminant condition (cf. [8]§ 13) that $\tau_1, \tau_2, \tau_3$ are on a line implies also that $\tau_4, \tau_5, \tau_6$ are on a line. Then we find as in (*d*) that the even function $\Theta_{123}(d) = \Theta_{456}(d)$ vanishes. The corresponding condition on the associated set $S_6^3$, $P$ of the set $T_7^2$ is that $P$ is on a plane with $s_1, s_2, s_3$ and $s_4, s_5, s_6$ or that $P$ is on a quadric section of $W_2$ by the pair of planes $\overline{s_1\, s_2\, s_3}$, $\overline{s_4\, s_5\, s_6}$. On transforming this condition on $T_7^2$ by $J_0(O, \tau_1, \tau_2)$; on $S_6^3$, $P$ by $I_{12}$; and on $\Theta_{123}(d)$ by (7) we find a second type of discriminant condition which is satisfied when $\tau_3$ approaches coincidence with $O$; when $P$ approaches coincidence with $s_3$ in a direction on the quadric cone with node at $s_3$ and simple points at the other points of $S_6^3$, or $P$ is on the quadric section of $W_2$ by this cone; and finally when the odd function $\Theta_3(d)$ vanishes. In this fashion we conclude that the quadric sections of $W_2$ by quadrics on $S_6^3$ are represented by the plane sections of the surface $K_2$ obtained by setting the coordinates equal to four linearly independent theta squares, i. e. of the Kummer surface. If four points $\tau_1, \ldots, \tau_4$ are on a line, $d \equiv \pi_{56}$, one of the fifteen proper half periods, and the point $P$ of $W_2$ is on the line $\overline{s_5\, s_6}$. Finally if $P$ is on the cubic curve $N^3$ through $S_6^3$ the set $T_7^2$ is on a conic ([8]Theorem 10) and $H_2$ is this conic doubly covered while $d \equiv 0$. In fact $d \equiv 0$ implies that a pair of the $g_1^2$ are nodal parameters which can happen only for a doubly covered conic for which the pairs of $g_1^2$ are the doubly covered points and $O$ is any of these double points. This is of course the indeterminate case for the inversion problem.

When $p = 3$ the four linearly independent quadrics on $S_6^3$ above are replaced by $8 = 2^p$ linearly independent cubic spreads with nodes at $S_8^5$ and containing the norm curve $N^5$ on $S_8^5$; and these map $W_3$ on the generalized Kummer $K_3$ of hyperelliptic type. The $2^{2p} = 64$ singular tangent spaces arise from the 8 cubic spreads (cones) with respectively a triple point at one of the eight points of $S_8^5$ and double curve $N^5$; and the 56 degenerate cubic spreads which break up into an $S_4$ on five of the eight points and a quadric on these five with nodes at the other three. These are connected with the 64 theta squares as above.

(*f*). Naturally we must have, in order to obtain specific analytic results such as (2), some analytic method for the hyperspace cases. Since the given equation defines in $S_{2p-1}$ the set $S_{2p+2}^{2p-1}$ and the norm curve $N^{2p-1}$ on it, or if one prefers the $N^{2p-1}$ with the set $S_{2p+2}^{2p-1}$ on it, it is convenient to refer the points of the space to this norm curve. The case $p=2$ is perhaps too special to be fairly illustrative, but I cannot go into the higher cases here. Let the parameters $t$ of the six points of $S_6^3$ on $N^3$ be given by the binary sextic $(\alpha t)^6=0$; and let the three osculating planes of $N^3$ which meet in a point $a$ of $S_3$ be given by the binary cubic

$$(at)^3=(a't)^3=\cdots=0.$$

Then, if

$$(\beta t)^2=\beta_0 t_1^2+2\beta_1 t_1 t_2+\beta_2 t_2^2$$

is an arbitrary quadratic, $\beta_3$ an arbitrary constant, we have, in

$$\beta_3(\alpha a)^3(\alpha a')^3+(aa')^2(a\beta)(a'\beta)=0, \tag{8}$$

for variable $\beta_3, \beta_0, \beta_1, \beta_2$ the web of quadrics on $S_6^3$ in variables $a_0, a_1, a_2, a_3$, the coefficients of the variable cubic $(at)^3$. The polarized form of (8) in variables $a, b$ is

$$\beta_3(\alpha a)^3(\alpha b)^3+(ab)^2(a\beta)(b\beta)=0. \tag{9}$$

If we assume that the quadric (8) has a node at $a$, then in (9) the coefficients of $b_0, b_1, b_2, b_3$ vanish giving rise to four equations bilinear in $a$ and $\beta$. If from these we eliminate the $\beta$'s we have the quartic equation in variables $a$ of the Weddle surface $W_2$. If, however, we eliminate the $a$'s we have the quartic equation in variables $\beta$ of the Kummer surface $K_2$ as an envelope. H. F. Baker has given these equations as four-row determinants ([9]p. 65 and p. 56). Baker remarks ([9]p. 60) that $K$ is a simultaneous invariant of the quadratic $(\beta t)^2$ and the sextic $(\alpha t)^6$. Strictly speaking this is not so, because of the non-homogeneity in the coefficients. However $K$ can be written

$$K=K_0\beta_3^4+K_1\beta_3^3+K_2\beta_3^2+K_3\beta_3+K_4, \tag{10}$$

where the $K_i$ are such simultaneous invariants. From the degrees and to some extent from the form we infer at once that these coefficients must be numerical multiples of respectively: the catalecticant of $(\alpha t)^6$; the apolarity invariant of $(\beta t)^2$ and the quadratic covariant of $(\alpha t)^6$ of degree 3; the apolarity invariant of $[(\beta t)^2]^2$ and $(\alpha\alpha')^4(\alpha t)^2(\alpha' t)^2$, together with the product of $(\beta\beta')^2$ and $(\alpha\alpha')^6$; the apolarity invariant of $(\alpha t)^6$ and $[(\beta t)^2]^3$; and $[(\beta\beta')^2]^2$. Similarly the surface $W$ is a simultaneous invariant of $(\alpha t)^6$ and $(at)^3$ which from the degrees must evidently be the apolarity invariant of $(\alpha t)^6$ with the product of $(at)^3$ and its cubic covariant, i. e.

$$W \equiv (aa')^2(a'a'')(a\alpha)(a''\alpha)^2(a'''\alpha)^3 = 0. \tag{11}$$

($g$). The extension of this method to further cases constitutes a remarkably neat application of many phases of the theory of binary forms. Let me give one illustration. For $p = 3$ and given $S_8^5$, a point in $S_5$ is determined by a binary quintic $(at)^5 = (a't)^5 = \cdots$. The catalecticant of this quintic,

$$(aa')^2(aa'')^2(a'a'')^2(at)(a't)(a''t) = 0,$$

determines a one-parameter ($t$) system of cubic spreads with $N^5$ as a double curve and triple point at $t$. Thus for $t = t_1, \ldots, t_8$ we have respectively the eight cubic cones mentioned in 5. The existence of this system leads to the following theorem.

THEOREM. *The Kummer* 3-*way in* $S_7$ *with* 64 *singular points and* 64 *singular tangent spaces has in the hyperelliptic case an additional configuration of* 64 $S_3$*'s each with the following property: An* $S_3$ *contains* 8 *singular points and is contained in* 8 *singular tangent spaces. The eight points lie on a cubic curve in the* $S_3$ *with parameters* $t_1, \ldots, t_8$ *and they lie three at a time on the remaining* 56 *singular tangent spaces.*

We observe then that the generalization of the conics in the singular tangent planes of the Kummer $K_2$ is in part the 2-ways in the singular tangent sections of $K_3$, and in part the cubic curves in the 64 $S_3$'s, each curve containing an intermediate set $R_6^8$.

A fuller development of these ideas leads quite naturally to the proof of the formula (2) which connects the invariants (B) and the theta fourth powers. To these modular relations I shall briefly return.

4. *The Modular Spreads Associated with the Invariants* (A) *and* (B). (*a*). The number of invariants (B) is $\frac{1}{2}\binom{2p+2}{p+1}$. I have shown ([4]I(10)) that only $\nu = \frac{1}{p+2}\binom{2p+2}{p+1}$ of these are linearly independent. Similarly the number of invariants (A) is $(2p+2)!/(p+1)!\,2^{p+1}$. I shall show in a forthcoming paper which will elaborate the indications presented here that the number of linearly independent invariants (A) is likewise $\nu$. Thus the invariants (A) and (B) lie in linear systems of the same dimension $\nu-1$ and it is sufficient to know the values of $\nu$ independent invariants of either system in order to obtain all of them. They are of course subject also to many relations of higher degree. By means of either linear system the totality of birationally distinct binary $(2p+2)$-ics is mapped upon the points of a "modular spread" $M_{2p-1}$ of dimension $2p-1$ in $S_{\nu-1}$.

(*b*). For the invariants (A) we secure the mapping by fixing all but one of the points of the set $S_{2p+2}^{2p-1}$ at a convenient "base" and allowing the last one to vary over $S_{2p-1}$. Then the invariants (A) become spreads of order $p$ with $(p-1)$-fold points at the fixed base points and this linear system maps the space $S_{2p-1}$ upon $M_{2p-1}$. To the permutation group of the roots of the underlying binary $(2p+2)$-ic there corresponds, in $S_{\nu-1}$, a collineation group of order $(2p+2)!$, $C_{n!}$, under which $M_{2p-1}$ is invariant; in $S_{2p-1}$, however, the Moore cross-ratio group which is generated by $(2p+1)!$ permutations of the fixed base points and an additional Cremona involution with $2p$ of the fixed base points as $F$-points and the remaining one a fixed point (cf. for the sextic [10]§ 1).

For the invariants (B) attached to the set $R^{p}_{2p+2}$ we select $\nu$ independent ones as coordinates in $S_{\nu-1}$ and then find that the permutation of the roots or of the points of the set $R$ leads likewise to a collineation group $C'_{n!}$ in $S_{\nu-1}$. It is the obvious thing to make such a selection of coordinates that the collineation groups $C_{n!}$ and $C'_{n!}$ attached respectively to the systems (A) and (B) shall coincide. Furthermore in order to distinguish between the two mappings we map by the system (A) upon the points of $S_{\nu-1}$ and by the system (B) upon the $S_{\nu-2}$'s in $S_{\nu-1}$.

(*c*). For the sextic, $\nu = 5$ and the modular spread $M_3$ in $S_4$ is a cubic three-way, the map of $S_3$ by quadrics on 5 points. By proper choice of the quadrics $M_3$ takes the form $\sum_{i=1}^{6} x_i^3 = 0$ where $\sum_{i=1}^{6} x_i = 0$. The collineation $C_{6!}$ is then merely the permutations of the six coordinates $x_i$. By proper choice of the invariants (B) the planar sets of 6 points on a conic, $R^2_6$, are mapped upon the $S_3$'s of $S_4$, $u_1, \ldots, u_6$ where $\sum_{i=1}^{6} u_i = 0$ in such a way that the relation

$$\left(\sum u_1 u_2\right)^2 - 4\left(\sum u_1 u_2 u_3 u_4\right) = 0$$

is satisfied. Here then the mapping is such that the invariants (A), (B) of the same binary sextic give rise to point and tangent space at the point of the *same* modular spread ([10](33)).

For the octavic, $\nu = 14$, and the geometry is hardly suitable for hasty exposition. Thus the modular spread attached to the invariants (A) is an $M_5^{40}$ of order 40 and dimension 5 in $S_{13}$. We have, however, in the formula

$$\varepsilon(ij)(kl)(mn)(op) = \vartheta^2_{ijkl}\vartheta^2_{ijmn}\vartheta^2_{ijop}, \tag{12}$$

where $\varepsilon$ is a properly chosen sign, the expression of the invariants (A) as uniform modular functions which are in immediate algebraic relation with the invariants (B). For detailed study the tactical relations exhibited by E. H. Moore[11] are most advantageous.

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