The author remarks in his preface that the material has been so arranged that "the text in large type ... may be read by itself". Yet we find on page 34 (large type) references to equations (7) and (8), which are in small type. All these, however, are unimportant oversights, and the reviewer turns from this work with the feeling that there is in it a wealth of valuable information on practically every phase of the theory of functionals, with many suggestions for its future development. The bibliographical list that heads each chapter is a feature of not inconsiderable value.

I. A. BARNETT

JUNG ON ALGEBRAIC FUNCTIONS

Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen, by Heinrich W. E. Jung. Berlin, Walter de Gruyter, 1923. 246 pp.

The three great paths in the study of the algebraic functions of a complex variable—the geometric, the analytic, and the arithmetic—have as a common starting point a single algebraic equation, $f(x,y) = 0$. The traveler on one of the roads, once away from the point of departure, is often far out of hailing distance from those on the other paths; yet he is at times agreeably surprised to find he has reached the same point as they. At such times there will be a sign-post telling him and his fellow-climbers that they have reached the Riemann-Roch Theorem, it may be, or the Lückensatz of Weierstrass. Whatever the point to which the various paths converge, it is almost certain to be concerned with *genus*, or *deficiency*, if another language is used.

Multiplicity of dialects is, indeed, characteristic of the study in question. Not only has each path its own vocabulary, but the arithmetic path, with which we are chiefly concerned here, has no single valid language.* In Jung's book, for instance, we miss the mention of *Ring, Führer, Ideal, Modul, Integrabilitätsbereich, Polygon,* although most of the concepts named find a place. On the other hand, certain terms are borrowed from algebraic geometry, in particular, *canonical class* (corresponding to the canonical series) and *adjoint functions* (corresponding to adjoint curves).

* For a comparison of the content, and to some extent of the language, of the various theories, see Emmy Noether, *Die arithmetische Theorie der algebraischen Funktionen einer Veränderlichen, in ihrer Beziehung zu den übrigen Theorien und zu der Zahlkörpertheorie*, JAHRESBERICHT DER D. MATH.-VEREINIGUNG, vol. 28 (1919).

Although other paths to the study of algebraic functions have probably greater beauty and historical importance, the arithmetic approach can justly claim to be the most logical one. For the arithmetic theory alone concerns itself primarily with the functions, not with curves or integrals. And Jung's book does much to increase the esthetic claims of his route.

Hensel and Landsberg's scholarly treatise on the arithmetic theory* being the only predecessor of Jung's *Einleitung,* a comparison of the two is valuable. The newer book is much the shorter. Brevity is obtained by assuming somewhat greater mathematical knowledge at the outset, and by restricting the scope of the study. The reader is expected to be acquainted with much of the theory of functions of a complex variable, in particular, with facts on development of functions in power series and on the inversion of functions so developed, and with essential singularities. Riemann surfaces are, however, explained from the beginning. Knowledge is assumed of the Lagrange interpolation formula, a little of the theory of matrices, and both point and line (homogeneous) coordinates in the plane.

Chapter I deals with an algebraic function of x defined by a single equation and with its behavior (i. e., development in power-series) in the neighborhood of any point. An important concept introduced here is that of the *Einheit* (we may call it an E-function in English), which, at a given point, is finite but not zero.

Chapter II gives methods for computing the power-series for an algebraic function at any point. Puiseux's diagram is clearly and fully explained. We could wish, in both these chapters, for more attention to convergence proofs.

The next chapter is devoted to an admirable exposition of the Riemann surface. Particularly clear are the figures, the account of the identification of branches, and the summary.

The preliminary apparatus having thus been set up, Jung proceeds, in Chapter IV, to the algebraic body (xy); this consists of all rational functions of x and of y, the variables being connected by the algebraic equation $f(x, y) = 0$. The essential properties of the functions of the body are then developed: unique definition on the Riemann surface, equality of numbers of zeros and of poles, representation in power series, etc. The residue of a differential $\eta d\xi$ appears in this chapter (η and ξ are functions of the body), as well as a few theorems preparatory to the reappearance of the residue in the chapter on integrals.

Chapter V is entitled Birational Transformations. There is in the arithmetic treatment no thought of the (x, y) plane, no occasion, accord-

* *Theorie der algebraischen Funktionen einer Variabeln und ihre Anwendung auf algebraische Kurven und Abelsche Integrale.* Teubner, Leipzig, 1902.

ingly, for one-to-one transformations of that plane. The chapter centers about the theorem that, if ξ is a variable of the body (xy), there exists another variable η of (xy), such that the bodies (xy) and $(\xi\eta)$ are identical.

In Chapter VI we come to what is probably the most characteristic tool of the arithmetic theory of algebraic functions, the *divisor*. Strangely enough, one is never told *what* a divisor is. To every place (Stelle) $\mathfrak{p}$ of the algebraic body "is assigned" a prime factor with the same name $\mathfrak{p}$. The essence of a prime factor is defined as follows. If t is a function with a zero of lowest possible order at $\mathfrak{p}$, and if a function R is the product of t^λ and an E-function at $\mathfrak{p}$, then R is said to be *divisible* by $\mathfrak{p}^\lambda$. A prime factor is, one might say, the minimum amount of zeroness at a point. A divisor is a product of integral powers (positive, negative, or zero) of prime factors. A divisor is determined, for instance, by the zeros and poles of a function. All divisors are grouped in classes; such a class consists of products of a single divisor and of all divisors of functions. The latter constitute, of course, a class by themselves, the principal class; while differentials, $\eta d\xi$, furnish another, the differential or canonical class. Linear independence of integral divisors is ingeniously defined, and the number of linearly independent integral divisors in a class is called its dimension. It is a misfortune, probably irremediable, that this number is one more than the dimension of the equivalent linear series of groups of points on a curve, — that two languages use the same word in senses just different enough to be misleading. In this chapter we notice an uncertainty as to what an α-fold branch point is. On page 71, line 1, a three fold branch point is one where four sheets join; yet on page 73, line 24, α sheets dance around an α-fold branch point. "Exactly divisible", a phrase introduced on page 79, ought to be defined. In this chapter the genus enters, defined by the number of sheets and by the sum of orders of branch points.

Chapter VII is devoted to matrices. It is hard to see why matrices should be called "systems" throughout this chapter, and developed from the beginning, when Jung will later (p. 154) not only talk unconcernedly of a matrix, but assume that the reader knows facts about it not mentioned in Chapter VII. The treatment is, at any rate, very satisfactory. The introduction of "multiplicative" and "additive" matrices for producing, on multiplication, desired changes in given matrices, is decidedly ingenious. Other important types are the "Einheiten", which we might now translate as "E-matrices". They are square matrices of rational functions of x, whose determinants behave as E-functions, either at all points, or at a single one. It seems unfortunate that matrices should be walled in indiscriminately by parentheses and braces. As parentheses are also used by Jung to indicate a class of divisors, braces to indicate its dimension, the double bar before and after would certainly be better, if the printer could be persuaded.

The eighth chapter treats the problem of determining all multiples of a given divisor; in the language of other writers the totality of such multiples is an "ideal". The solution is reduced to the determination of a "base" for the functions divisible by the divisor — a group of functions such that, in linear combination with rational coefficients, they generate the body (xy), whereas with integral rational coefficients they generate the multiples of the divisor. The task of showing that such a base can always be found is a strenuous one, the chapter correspondingly long and difficult; but the work is done in masterly fashion. There are, at the end, two sections on complementary bases: pairs of bases, such that the two matrices of the functions and of their conjugates are complementary.

The final chapter of the purely arithmetic theory is devoted to the Riemann-Roch Theorem. As here given, it is concerned with the dimensions and orders of two "Ergänzungsklassen", i. e., classes such that a divisor of one and a divisor of the other have as product a divisor of the canonical class. The central position of this theorem arises from the information it furnishes concerning the existence of functions with given poles and zeros. Several consequences of the theorem for bodies of genera 0, 1, and 2 are deduced, and the chapter ends with the introduction of Weierstrass points.

Two chapters are now devoted to the geometry of the curve $f(x, y) = 0$. In one of them, non-homogeneous coordinates are used, and in the other the change to homogeneous coordinates is made. Many headings of sections, such as multiple points, form of real branches, number of intersections, Hessian curve, Plücker formulas, are those found in any book on projective geometry. The treatment, however, using the divisor on almost every page, gives to familiar facts unfamiliar clothing. Plücker's formulas, for example, are equations between orders of divisors.

Just as there is, in the chapters on geometry, an adequate transition to the geometric theory, so we find, in the last two chapters, an introduction to the integral theory. They contain, respectively, a treatment of the analysis situs of Riemann surfaces, and a short discussion of the three kinds of abelian integrals. The writing is clear and well-balanced; yet one feels Jung's realization that his chief contributions to the mathematical reading public had been made earlier in the book.

If I had mentioned the book's chief claim to distinction in any single chapter, I should have needed to repeat it in nearly all. About one-fourth of the space is devoted to examples, carefully chosen and clearly worked out at more or less length. The student who merely checks the results given will have a better knowledge of the real meaning of the theory than one who reads, however carefully, the scholarly book of Hensel and Landsberg. Often there are gaps and suggestions enough to furnish the reader with material for real study; and every example worked out can, of course, suggest other problems to be solved ab initio.

Among the misprints and slips which I have noted, the correction of the following will be of importance to the reader.

PAGE	LINE	FORMULA	FOR	READ
9	30		des in ihm liegenden	eines in ihm liegenden
22	13		keine der anderen Strecken	alle anderen Strecken
49	5		*arg* x	*arg* $(x-a)$
72	5		wo t eine Einheit	wo $E(t)$ eine Einheit
80	8		Kap. VIII	Kap. IX
102		9	$g(x) \pm h(x)y$	$g(x)+h(x)y$
106	7		(22)	(18)
165	3rd row, 2nd matrix, middle of page,		0 1 0 0	0 0 1 0
173		5	$\frac{\partial f}{\partial x}=\frac{\mathfrak{d}\mathfrak{z}_y}{\mathfrak{l}^{-2}\mathfrak{m}^m}$	$\frac{\partial f}{\partial x}=\frac{\mathfrak{d}\mathfrak{z}_y}{\mathfrak{l}^{l-2}\mathfrak{m}^m}$
186	27		die p_k definierende	die p_1 definierende
191	26		Ebenenkoordinaten	Linienkoordinaten
195		30	$-\mathfrak{a}_3 d\mathfrak{a}_1$	$-\mathfrak{a}_3 d\mathfrak{a}_2$
196		38 }	$\mathfrak{d}$	—
197		39 }		$\mathfrak{d}$
211	24		$\lim \frac{\xi}{\eta}=-4$	$\lim \frac{\eta}{\xi}=-4$
235	19		$=\lambda$	$=\lambda+p$
238		15	ρ	p

Jung's book is adequately and clearly illustrated, fully and accurately indexed. His style is good. The prospective foreign reader will be particularly encouraged by the fact that the average sentence is but two lines long. It is to be hoped that this book will have many readers in America, in spite of the obstacles which our educators' continuing slight of the German language puts in the way of our next generation's progress in science. A moral may be learned from the fact that Germans have almost a monopoly of the work published on the arithmetic theory of algebraic functions.

All in all, we can say that Jung has chosen a task well worth doing, and has performed it excellently.

E. S. Allen