EVANS' COLLOQUIUM LECTURES

The Cambridge Colloquium Lectures, 1916. *Part* I: *Functionals and their Applications. Selected Topics, including Integral Equations.* By G. C. Evans. New York, American Mathematical Society, 1918. 136 pp.

The theory of functionals has been in the foreground so much in the last two decades, particularly in the domain of integral equations, that a colloquium on this subject was very timely. The theory of integral equations, however, bears the same relation to the whole theory of functionals as the theory of algebraic equations does to the entire domain of the theory of functions. In fact, so rapid has been the development of the theory of functionals that Paul Lévy has recently devoted a book of nearly four hundred fifty pages to the study of functional analysis alone, leaving aside entirely the theory of complex functionals and touching but slightly the theory of integral and integro-differential equations and a number of other subjects that would ordinarily be included as topics in the theory of functionals. It can be seen from this what a difficult, and what seems to the reviewer an almost impossible, task it would be to essay a clear presentation of almost every domain of this difficult subject in the space of less than one hundred fifty pages. This was the task that Professor Evans set himself.

The author has divided his work into five lectures. The first lecture takes up general considerations of a functional, such as definitions of continuity, Volterra derivatives, and additive and non-additive functionals of plane curves. A rather interesting connection is noted between additive functionals and functions of point sets, viz., "an additive continuous functional of finite variation has a finite derivative (in the restricted sense) at all points except possibly those of a set of measure zero". Another interesting result is the extension of the law of the mean of the differential calculus. Functionals of space curves are then introduced, and the concept of the *flux of a functional,* due to Lévy, is defined, and is used extensively in the next lecture.

In studying the dependence of the Green's function on the boundary of the closed curve, Hadamard was led to an equation involving functional derivatives. These equations are the analogs of total and partial differential equations in n variables. Just as, in the latter, certain conditions of integrability are required, so here also analogous conditions must be satisfied. These are developed very elegantly by means of another concept due to Lévy, that of the *adjoint linear functional.* Various interesting applications are made of the integrability conditions, and, in particular, it is shown that Hadamard's equation is *completely integrable.*

The author now turns his attention to complex functionals, to which he devotes Lecture II. As far back as 1889 a considerable portion of this theory was developed by Volterra, who takes as a starting point the relation of *isogeneity*, which is the extension to functionals of curves in space of the relation that holds between two complex point functions on a surface. This condition of isogeneity is expressed in terms of the normal component to the curve of the vector flux of a functional defined in Lecture I. Although the author summarizes the properties of the linear vector function used in this lecture, it would be highly desirable for one to have some acquaintance with vector analysis before attempting to read it. In case the functionals involved are additive, the condition of isogeneity may be considerably simplified, since in this case the vector flux can be chosen as a vector point function independent of the curve. Important properties of the relation of isogeneity for additive functionals are given, and the analogs of Green's and Cauchy's theorems are proved. All this has been extended by Volterra to additive complex functionals whose arguments are r-dimensional hyperspaces immersed in n-space. For $r=n-2$, this is a direct generalization of the theory of complex functionals of curves in 3-space.

Although Lecture III has the caption "Implicit Functional Equations", this part of the work is essentially concerned with the study of the linear functional. This study must necessarily precede any existence theorem on implicit functional equations, since the differential, which has a natural origin here, is a linear functional. $T[\Phi]$ is said to be a *linear functional* of Φ if (1) it is a distributive functional of Φ, and (2) if it is a continuous functional of Φ, where Φ is allowed to range over the whole class of continuous functions. Various representations of such a linear functional have been given by Hadamard, F. Riesz, and Lebesgue, using as their means of representation the Riemann, the Stieltjes, and the Lebesgue integrals, respectively. Since, in the definition of a linear functional, one may change either the class of functions Φ, or the type of continuity, one may obtain two distinct extensions of the linear functionals. The first has been made by Fréchet, Lebesgue, Radon, and others. The second extension is to introduce higher orders of continuity. The importance of this concept seems to have been recognized first by Bliss,* who notes that if the calculus of variations is to be regarded as a chapter in the theory of maxima and minima of functionals, the definition of continuity of a functional must be extended in this way.

Fréchet's definition of a differential of a functional is now given, and a theorem on implicit functional equations due to Volterra is proved

* *A note on functions of lines*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 1 (1915), p. 173.

in which the differential is used. It must be pointed out, however, that Volterra uses a very special kind of differential. In fact, Volterra proved his result some time before Fréchet gave his formal definition.

Lecture IV opens with the remark that the most desirable approach to the treatment of Laplace's equation is to study, as Bôcher did, the equation

$$\int_c \frac{\partial u(x, y)}{\partial n} ds = 0.$$

This relation is supposed to hold for all circles within a given two-dimensional region, and the existence and continuity of u, u_x, u_y are assumed, instead of the existence of u_{xx}, u_{yy}, which have no physical significance. This lecture is practically devoted to the study of an equation which is a generalization of the one given above. Professor Evans calls such equations integro-differential equations of the Bôcher type. By defining the adjoint of such equations, the author is able to prove an extension of Green's theorem for them. He now subjects the equation to an arbitrary real point transformation, and finds that the equation is transformed into a similar one; and, furthermore, that a certain differential quadratic form $T(dx, dy)$ is a covariant of the transformation. The directions defined by $T(dx, dy) = 0$ are called *characteristic directions,* and the solutions of the equation are called *characteristic curves.* The characteristic directions may be real and distinct, real and coincident, or imaginary. The integro-differential equation is said to be hyperbolic, parabolic, or elliptic, respectively, just as in the theory of linear partial differential equations. Normal forms of these types are then obtained, and a rather detailed study of the parabolic type is made. The lecture closes with some remarks on the usual types of integro-differential equations, as originally given by Volterra.

Lecture V, the last of the series, gives an account of the various generalizations of the theory of integral equations. The lecture begins with a statement of some of the more important properties of Stieltjes' integrals, and an application of these to the study of a class of equations involving Stieltjes' integrals. An existence theorem of such equations is given, but with the hypotheses as stated, an extension of the Fredholm theory is not possible.*

In generalizing the theory of the linear integral equation, one may

* It might be stated here that such a generalization has recently been given by F. Riesz in a very important paper in the ACTA MATHEMATICA (vol. 41 (1916), p. 71), which unfortunately appeared just too late to be included in these lectures. He uses there the concept of *completely continuous linear functionals*; it was this notion that made the extension possible.

proceed along at least two distinct lines. In the first place, one may, with E. H. Moore, generalize the variables, the classes of functions, and the linear operations, in such a way that the methods of Fredholm apply for the extended concepts. Or one may proceed, as Volterra has done in his *Theory of Permutable Functions,* to build a calculus of composition, i. e., of the operations which produce the iterated kernels. He is able by this means to solve an extensive class of non-linear integral and integro-differential equations. These extensions Professor Evans takes up in the order mentioned, and it should be remarked that an exceptionally clear exposition of Moore's theory is given, and a number of important contacts with the classical theory are pointed out. He notes, for example, that when the general range P is the one-dimensional continuum, and the class M is the totality of continuous functions over this range, Moore's linear operation J reduces to the classical Stieltjes' integral.

In the presentation of Volterra's theory, Professor Evans makes a considerable advance over the treatment found in Volterra's *Leçons sur les Fonctions de Lignes.* In the first place, he combines the theories of permutable functions of the first and second kinds into a single theory. In the second place, he introduces notations and concepts that simplify materially the proofs given by Volterra. This theory of permutable functions leads immediately to a very interesting extension of every analytic function. This extension carries with it corresponding extension of addition theorems, moduli of periodicity, etc. One such generalized function, called the *Volterra transcendental,* plays an important role in a certain class of integro-differential equations.*

It can be seen from this outline what a multitude of topics has been discussed in these lectures. After reading this book, one feels that the author is profoundly at home with every phase of the theory of functionals, to which he himself has made many important contributions. Although a serious effort has been made to give a clear presentation of these subjects, one cannot help feeling that it might have been wiser to have included fewer topics and to have developed each of them in a more leisurely fashion.

Of the misprints, one may mention those which have been noted already by Professor Evans in this BULLETIN (vol. 25, p. 461) and the following. In the theorem at the top of page 7, change $F'[\varphi(x) \mid \xi]$ to $F'[\varphi_0(x) \mid \xi]$. The left member of formula (24′) on page 16 should read $g'[c \mid ABM]$ instead of $g[c \mid ABM]$. On page 69, the right member of formula (33) should read $F[\varphi + \Delta\varphi] - F[\varphi]$.

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* See Schlesinger, *Zur Theorie der linearen Integro-Differentialgleichungen,* JAHRESBERICHT DER VEREINIGUNG, vol. 24 (1915), pp. 84–125.

The author remarks in his preface that the material has been so arranged that "the text in large type ... may be read by itself". Yet we find on page 34 (large type) references to equations (7) and (8), which are in small type. All these, however, are unimportant oversights, and the reviewer turns from this work with the feeling that there is in it a wealth of valuable information on practically every phase of the theory of functionals, with many suggestions for its future development. The bibliographical list that heads each chapter is a feature of not inconsiderable value.

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JUNG ON ALGEBRAIC FUNCTIONS

Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen, by Heinrich W. E. Jung. Berlin, Walter de Gruyter, 1923. 246 pp.

The three great paths in the study of the algebraic functions of a complex variable—the geometric, the analytic, and the arithmetic—have as a common starting point a single algebraic equation, $f(x,y) = 0$. The traveler on one of the roads, once away from the point of departure, is often far out of hailing distance from those on the other paths; yet he is at times agreeably surprised to find he has reached the same point as they. At such times there will be a sign-post telling him and his fellow-climbers that they have reached the Riemann-Roch Theorem, it may be, or the Lückensatz of Weierstrass. Whatever the point to which the various paths converge, it is almost certain to be concerned with *genus*, or *deficiency*, if another language is used.

Multiplicity of dialects is, indeed, characteristic of the study in question. Not only has each path its own vocabulary, but the arithmetic path, with which we are chiefly concerned here, has no single valid language.* In Jung's book, for instance, we miss the mention of *Ring, Führer, Ideal, Modul, Integrabilitätsbereich, Polygon,* although most of the concepts named find a place. On the other hand, certain terms are borrowed from algebraic geometry, in particular, *canonical class* (corresponding to the canonical series) and *adjoint functions* (corresponding to adjoint curves).

* For a comparison of the content, and to some extent of the language, of the various theories, see Emmy Noether, *Die arithmetische Theorie der algebraischen Funktionen einer Veränderlichen, in ihrer Beziehung zu den übrigen Theorien und zu der Zahlkörpertheorie*, JAHRESBERICHT DER D. MATH.-VEREINIGUNG, vol. 28 (1919).