PROBLEMS IN INVOLUTORIAL TRANSFORMATIONS OF SPACE*

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1. *Introduction.* A report of great value and of general interest was presented to this Society at the Chicago meeting of April, 1922. While only a limited number had the opportunity to hear Professor Coble on that occasion, fortunately his message has reached a much wider public, by appearing in this BULLETIN (vol. 28 (1922), pp. 329–364). On account of the wider purpose there in hand, it was impossible to treat in detail all the many ramifications of the theory, or to show all their interrelations. My present purpose is to comment more fully on one narrow phase of this report, namely, that of involutions.

In the plane the problem is almost completely solved. It was shown by Bertini(1)† that there are four types to one of which every plane involutorial transformation can be reduced; they are the harmonic homology (H), the Geiser(2) (G) of order 8 with 7 triple points, the Bertini(1) (B) of order 17 with 8 six-fold points, and the Jonquières(3) (J) of order n with one fundamental point of order $n-1$ and $2n-2$ simple ones. Of these, all but the last are individual types, but (J) exists for every positive integral value of n. For special (J), JH is also an involution, but this can always be reduced to an H. Moreover, all these involutions are rational, that is, the pairs of conjugate points can be mapped rationally upon a plane (x') such that between (x') and the given plane (x) there exists an algebraic $(1, 2)$ point correspondence.

Thus, that associated with (H) may be expressed in the form $x_2'x_3 - x_3'x_2 = 0, \quad x_1'x_2x_3 - x_2'x_1^2 = 0$. The invariant

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†The numbers refer to the papers listed at the end of this article.

or coincident points constitute the line $x_1 = 0$. That associated with (G) has the form

$$x_1' C_1 + x_2' C_2 + x_3' C_3 = 0,$$
$$x_1' x_1 + x_2' x_2 + x_3' x_3 = 0,$$

in which each $C_i = 0$ is a general conic. Each cubic of the net formed by linear combinations of $x_i C_k - x_k C_i = 0$ is transformed into itself. The curve of invariant points is the jacobian of the net, $K_6 (r_1^2 \cdots r_7^2)$, in which r_i is a simple basis point. The curve of branch points in (x'), in $(1, 1)$ correspondence with K_6, is the general quartic L_4', which expresses the condition that the line in (x) touches the associated conic. Every line of (x) contains one pair of conjugate points. From this standpoint the system of the bitangents, the contact conics, cubics, etc. of L_4' can all be determined directly by elementary methods. Their two images in (x) are conjugate as to (G). That associated with (B) may be expressed by the equations

$$x_1' \varphi_1 + x_2' \varphi_2 = 0,$$
$$x_1' f + x_3' \varphi_1 (k_1 \varphi_1 + k_2 \varphi_2) = 0,$$

in which $\varphi_i = 0$ is a general cubic curve, and $f = 0$ is a proper sextic having eight of the points h_i common to $\varphi_1 = 0$, $\varphi_2 = 0$ for double points. The curve of invariant points is the jacobian of the system $\varphi_1 = 0$, $\varphi_2 = 0$, $f = 0$. It is of order 9, and has eight h_i as triple points, $K_9 (h_1^3 \cdots h_8^3)$. The curve of branch points is also a sextic L_6', having three branches touching each other at a common point. An arbitrary line in (x) contains four pairs of conjugate points; i. e., the simplest form of (B) is of class 4. Finally, the $(1, 2)$ correspondence of form (J) can be expressed by $x_1' x_2 - x_2' x_1 = 0$, $x_3' M_1 - u' M_2 = 0$, in which $M_i = 0$ is a curve of order m with an $(m-2)$-fold point at $(0, 0, 1)$; u' is linear in (x'). The jacobian is the curve of coincidences, a hyperelliptic curve having an $(n-2)$-fold point at $(0, 0, 1)$. This involution is of class zero. This is the easiest way to find and to classify the plane involutions of order 2; to find all possible ways in which the curves of a net can have two variable points of intersection. The question of reducibility or of the equivalence of two given cases is not more difficult by this method than by that of Bertini. The seven basis points r_i have for images in (x') seven bitangents of L_4'; a point of a bitangent has two images, one of which is the basis point r_i, and the other describes the cubic of the net having a node at r_i. The point and the nodal cubic are conjugate in the involution (G).

Indeed plane involutions of every order were later shown to be rational, by Castelnuovo(4). For particular positions of the basis points the types assume special forms, but they are all included in the same (or simpler) categories(5). An interesting case of (J) is that in which all the curves of the net have contact of the second order on each branch at the multiple point. Its equation is

$$\lambda_1(-u_{n-1}x_3+u_n)+\lambda_2 u_{n-1}x_1+\lambda_3 u_{n-1}x_2=0$$

in which u_i is binary in x_1, x_2 of order i. All the basis points are now coincident at (0, 0, 1).

The curves of coincident points characterize the type of the involution. Hence it follows that any surface mapable on a double plane is, or is not, rational according as its curve of branch points can be reduced to one of the above curves(6). Another possibility is that involutions may be compounded. This is particularly useful in studying irrational algebraic curves. A curve is hyperelliptic if it has a linear series g_2^1. Thus every hyperelliptic curve is invariant under at least one involution. But a curve may be cut by the curves of a pencil in such a way that each curve cuts more than one pair of conjugate points from the given one. Such involutions, possible only on curves of genus greater than one, are usually irrational; they exist for any given genus(7). No algebraic curve of genus greater than 1 can belong to an infinite group, either continuous(8) or discontinuous. An elliptic curve is invariant under an infinite series of operations which do not form a group. It has an infinite number of central involutions each with four fixed points, and three without fixed points. A rational curve is invariant under a three-parameter group, within which are ∞^2 involutions.

2. *Transformations in Space.* Of the linear transformations, only two are involutorial, the central and the axial involution. In the former we have a plane of invariant points, in the latter, two skew lines. The quadratic transformations may be defined as in the plane. They were all found in the early papers of Cayley(9), Cremona(10), and Noether(11). An elementary derivation is given by Snyder and Sisam(12). The involutorial forms are either perspective, such that every line of a bundle remains invariant, or the product of the perspective inversion and a central homology. A full discussion is given by Doehlemann(13). An application to the theory of electric images was given by Liouville(14). For the theory and the literature, see Doehlemann(13). One great application of the birational transformations of space is to the study of systems of curves lying on given surfaces. Thus, on the quadric, every algebraic curve can be expressed by means of an (a, b) correspondence, and only those plane curves which can be thus expressed are projections of space curves of the same order lying on a quadric. Since the surfaces of a homaloidal web are all rational, the transformation furnishes an immediate method of mapping the surface on a plane. On the other hand, this method does not furnish general criteria for determining whether a birational transformation is involutorial or not. Nearly all of the earlier involutions were found directly from particular geometric conditions imposed by the problem considered, and none have been obtained by following the procedure employed by Bertini for the plane involutions.

The surfaces of the system must satisfy two fundamental conditions; they must form a linear system with four independent surfaces; any three intersect in one variable point. Hence the surfaces must have points or curves common to all. But these criteria are common to all birational transformations.

3. *Rational Involutions.* Consider in space (x), a linear system of ∞^3 surfaces, a web, having the property that

any three not in a pencil intersect in two variable points. Let the web be $\sum a_i\varphi_i = 0$. By putting $x_i' = \varphi_i$ $(i = 1, 2, 3, 4)$ the surfaces can be mapped upon the double space (x'). This was first studied by De Paolis(15).

To the planes of (x') correspond the surfaces of the web $|\varphi|$, each of which remains invariant under the involution of associated points, images of the same point of (x'). To the lines of (x') correspond the curves (φ_i, φ_k), also invariant under I. These curves are either hyperelliptic or belong to sub-types of hyperelliptic curves, of genus 1 or 0. Each surface of the web $|\varphi|$ has then ∞^2 hyperelliptic curves, any two of which intersect in two variable points. The locus of points in (x') having the property that the two images in (x) coincide is the surface L' of branch points. The coincident points in (x) define a surface K, in (1, 1) correspondence with the points of L'. These surfaces may be replaced by curves or by points. The system of surfaces in (x'), images of the planes of (x), do not form a linear system, but have certain points and curves in common which are fundamental for the correspondence. A fundamental point P' may have one or both its images in (x) fundamental(16). If its images are a point P and a curve p, then P is a fundamental point of the associated involution I in (x). If both images are fundamental, the associated curve in (x) may be fundamental in different ways.

A complete enumeration of rational involutions would involve the various webs of surfaces with two variable intersections. While these are of course infinite, it is not known to what extent they can be reduced to a small number of families. Thus, in the plane, if we exclude the hyperelliptic curves with common multiple points there are only two nets of curves with two variable points, one of genus 1 and the other of genus 2. The equations concerning a Cremona net are then sufficient to complete the enumeration. When the point P' describes a locus (curve or surface), the images P_1, P_2 will each describe a locus. These two loci may coincide, or if distinct, they are conjugate in I. The necessary and sufficient condition that the two loci p_1, p_2 are distinct is that p' touch L' at every non-fundamental point(17). From this standpoint all the systems of bitangent planes, of contact quadrics, cubics, etc. of the general L' can be found immediately without the use of transcendental methods. In particular the system of contact surfaces of the Kummer surface can be derived from the web of quadrics through 6 points. The numerous formulas derived by De Paolis are not in the main of greatest usefulness on account of being expressed in too many unknowns. When the web is regular, and the basis elements independent, the Riemann-Roch theorem for surfaces,

$$r = p_a + n - \pi + 1,$$

applies, in which r, the dimension of the system of curves in which a fixed surface of the system is cut by the other surfaces of the web, is 2, n is the number of variable intersections of two curves of the system, and π is the genus of the variable curve. Hence this genus exceeds the arithmetic genus of the general surface of the web by unity. Since each curve is hyperelliptic, the number of coincidences is $2\pi + 2$, hence the order of L' is also $2\pi + 2$.

The general discussion of De Paolis was followed by Schoute(18), who confined his discussion to simple cases. Reye(19) gave a detailed synthetic study of the web defined by quadrics through 6 points, which has been treated by a number of writers since. Hudson(20) developed it from the dual of the previous standpoint; Eberhard(21) followed various systems of curves on the surface, to which I added several(22).

An involution is determined on a quartic surface with five basis points by the space cubics through these and a point P, but this may not be birational for all of space. In the case of the Weddle surface the points P, P' are collinear with the sixth node. Other cases are treated by Sturm(23), by Marletta(24), and by Baldus(16). The latter also discussed (1, 2) correspondences between irrational ruled surfaces, in which fundamental elements appear which are not found in planar involutions. A particular (1, 2) correspondence associated with a net of cubics through a space quartic curve γ_4 and five coplanar basis points was studied by Pieri(25), primarily for the properties of L'. An interesting property appears here, as the variable curves are of genus 2. The system of curves is irregular since the basis points are not independent. Three cases cited by Pieri as belonging to distinct types are now known to be reducible to particular cases of the types considered by him. Another important feature is the appearance of an irrational fundamental curve. In the plane the image of a fundamental point is always a rational curve of order equal to the multiplicity of the point. But the net of surfaces of the web which pass through an arbitrary point of the plane π all pass through the cubic curve of the pencil determined by this point, the basis points A, and the four points of intersection of π and γ_4. The image of this cubic in (x') is a point, and the point describes a straight line as the cubic curve describes the pencil. The other image of P' is a point R in (x) which depends upon how P' is approached. To an arbitrary plane through P' corresponds a linear g_2^1 on the cubic curve. This property accentuates the difficulty of defining a type, as an irrational curve may be transformed into a point.

Another web of cubics was considered by Romano(26), that having three skew lines and four points for basis elements. This presents six fundamental lines of the second kind, such that in the associated involution the image of any point on one of them is the whole line passing through it.*

In every case, when simple basis points of the web appear, the image of each in (x') is a tangent plane to L'; the conjugate, in the associated involution, of the basis point is the

*In the later discussion of the same case by Sharpe and Snyder, TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 56, these lines are overlooked, so that the characteristics of the involution as there given are incomplete. Similar omissions occur in the later types defined by a web of cubic surfaces. Romano's paper was not known to the authors at that time.

surface of the web having a double point at that point. When the web has a simple basis curve, each point of the curve goes into a straight line in (x'), and the conjugate in I is a rational curve. These curves generate a surface, conjugate of the given curve, which is a multiple basis curve in the involution.

But when the section of the surfaces of a web by a fixed surface is completely accounted for by fundamental elements, various paradoxes may appear. This has not been completely worked out, especially when the section is made on an irrational non-ruled surface.

The problem of the existence of these webs depends upon formulas of postulation involving contact of higher order at multiple points. In the original memoir of Noether(27) the cases are excluded in which the tangent cones to the surfaces of the web at a multiple basis point are composite, and contacts of different sheets along branches of the curve is not considered. Notwithstanding the recent valuable additions made by Hudson(28), by Severi(29), and by Tummarello(30), little progress can be made in the problem of reducibility and the determination of the equivalence of two given involutions until more is known on this subject.

When all the points of intersection of a line or a curve with surfaces of the web are fixed, or are on fixed basis curves, the line is called a parasite by De Paolis. Since a net of surfaces pass through it, its image in (x') is a point. If the given curve belongs to a linear system, each curve of which is a parasite, the image point describes a straight line.

A satellite is a line or point of contact which insists on coming in, unbidden, when certain conditions are imposed. Consider the conical surface having a cuspidal edge $x_1^2x_3 = x_2^3$, and the rational cone $x_1x_3^{n-1} + Ax_2^kx_3^{n-k} + \cdots = 0$ with undetermined coefficients. The number of lines of contact approaching coincidence with $x_1 = 0$, $x_2 = 0$, in the plane $x_1 = 0$ is always an odd number. If we impose the condition that one more line be counted, a second insists on accompanying it. This is true for a cone having any superlinear branch.

Satellites and parasitic branches will have to be considered in view of fundamental elements of new and as yet little understood forms. The explanation given by De Paolis(15) is incorrect, and the relation between a line and its parasite may violate the theorem of order and multiplicity, as has recently been shown by Montesano(31) and by Tummarello(30).

4. *Involutions by Means of* (1, 2) *Correspondences.* The first comprehensive study of the (1, 2) correspondence from the standpoint of classification of associated involutions was made by Sharpe and me(32). After collecting the necessary formulas of postulation and equivalence we applied them to webs of surfaces of orders 3, 4, and 5. Two kinds of restrictions were made at the outset. Surfaces with higher point singularities, including monoids, were excluded, and the basis elements were assumed to be independent. The first restriction was deliberate. When the surfaces are monoids or, more generally, rational, then each surface of the web can be mapped birationally upon the plane, and the curves of intersection with the other surfaces can be treated by means of plane involutions. The two extreme cases of monoids, those having one non-composite basis curve, and those having a fixed tangent cone at the vertex have been determined since, but the results are not yet published. The former are not monoidal transformations, and most of them cannot be reduced to the monoidal type. Those of the other form are always monoidal, of a very particular type. The other restriction was less deliberate. In the plane the basis elements are independent, and the fundamental curves are uniquely fixed by the position and multiplicity of the basis points through which they are to pass. If the elements are chosen in a particular way, for example three collinear points among the seven which define a Geiser net, the resulting involution can be reduced to a simpler form in which the elements are independent.

Within these limitations, the webs of cubic, quartic and quintic surfaces have been determined. Among them appear the focal surfaces of all the line congruences of order 2 and class 2 to 7, and several related surfaces, together making an unbroken sequence. Incidentally, configurations of curves on these surfaces can be determined much more directly by this method than by those by which they were first discovered. The category also includes surfaces invariant under infinite discontinuous birational groups, both those known previously, as the Kummer surface and the Fano surface, and new surfaces not previously studied.

What was not accomplished was to determine how many of the involutions appeared more than once in a transformed form. It was proved, for example, that there are different involutions having a Kummer surface for surface of branch points L' in (x'). Each focal surface of a line congruence of order 2 appears at least twice. Of course one will meet, in the study of webs of higher order, all the involutions associated with simpler webs, which can be transformed birationally into webs of the given order. Thus, Osborn(33) has studied the webs of order 6, and has refound many of the existing types. The webs of singular cubics have recently been studied by C. Moffa(34) by the method of plane mapping without the use of the double space, and the results in the other webs confirmed by the same method. Miss Moffa shows that fundamental lines of the second kind must exist in every case. This paper, together with the correction to the result obtained by Pieri(25), completes the list of involutions which leave each surface of a web of cubics invariant.

5. *Bases of Classification.* Another scheme of classification is to divide the involutions into families, expressed as a function of the order n, like the Jonquières forms in the plane. The first attempt of this kind was made by De Paolis(35), who classified all the forms that leave a line congruence of order one invariant. The lines may belong to a bundle, or meet a line and a rational curve of order m, with $m-1$ points on the line, or be the bisecants of a cubic curve. The second and third cases are treated exhaustively, but from the standpoint of classification the results are of little

importance, as all of them can be reduced to the monoidal form. The involution of order 7 defined by a web of quadrics through six basis points belongs in this category. Most of its properties are much more easily obtained, however, in the form of the web of quadrics, than expressed in terms of the nodal projection of the Kummer surface upon itself.

Another important family was mentioned by Noether(36) and more fully treated by Montesano(37). In this family the images of the planes of space are surfaces of order n with a common $(n-2)$-fold line and a basis curve of order $3n-4$. This will be discussed later. Another family, also discussed by Montesano(38), consists of surfaces of order $2n+1$, having as basis elements a space elliptic quartic to multiplicity n, and $2n$ of its bisecants for simple lines. This case was also previously cited by Noether(39). It includes a number of particular cases previously mentioned by others. Thus, the type discussed by Pieri(25) is this one when $n=5$. The case in which the conjugates of planes are surfaces of order n with a fixed $(n-1)$-fold line was also studied by Montesano(40).

A number of families of involutions, generalizations of the preceding, were found by Sharpe and me(17), the results of which can perhaps be best expressed by the theorem: There exists an involution of order $12n+5$ having a fundamental quartic γ_4 of genus 1 to multiplicity $6n-1$ and a fundamental curve β_{6n+1}, genus $9n-3$, to multiplicity 4, meeting γ_4 in $12n-4$ points. It has also $12n+16$ simple fundamental lines meeting γ_4, β_{6n+1} each twice and $6n-2$ fundamental double conics meeting γ_4, β_{6n+1} each in 4 points. In symbols,

$$s_1 \sim s_{12n+5}: \gamma_4^{6n-1} + \beta_{6n+1}^4 + (12n+16)u_1 + (6n-2)u_2^2,$$

$$[\gamma_4, \beta_{6n+1}] = 12n-4; \ [u_i, \gamma_4] = [u_i, \beta_{6n+1}] = 2.$$

Similarly,

$$s_1 \sim s_{4n+9}: \gamma_1^{4n+1} + \beta_{5n+5}^3 + (15n+10)u_1 + 2u_2^2,$$

$$s_1 \sim s_{6n+9}: \gamma_1^{2n+6} + \beta_8^{2n+1} + \beta_{7n}^2,$$

$$s_1 \sim s_{6n+17}: \gamma_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}^2,$$

$$s_1 \sim s_{6n+10}: \gamma_2^{2(n-1)} + \gamma_7^{2n+2} + \beta_{7n-3}^2.$$

These various types are not reducible to each other.

6. *Monoidal Transformations.* The monoidal transformations were discussed by De Paolis(41) and the product of the perspective monoidal and a harmonic homology was treated by Martinetti(42), who thought there could be no other non-perspective types. All the forms were derived synthetically by Montesano(43), who showed that every type could be reduced to one in which the bundle of lines through the vertex remains invariant, and that the lines of the bundle are either invariant (perspective type) or are interchanged in pairs according to one of the four ternary involutions H, G, B, J. Of these, the latter contains two parameters, the order n of J, and the order m of the perspective monoid. The equations of all the types were derived by me(44).

Let $O = (0, 0, 0, 1)$ be the vertex of the monoid. Since the bundle of lines O is transformed into itself, and is involutorial, it must be either the identity, or H, G, B, or J. In the first case, called the perspective, the conjugate of any point P is on OP and is harmonic with regard to P and the pair of points in which OP meets a fixed surface

$$u_{n-2}x_4^2 + 2u_{n-1}x_4 + u_n = 0,$$

u_i being ternary in x_1, x_2, x_3 of degree i.

The equations of the transformation are

$$rx_i' = x_i,\ i = 1, 2, 3; \quad rx_4' = \frac{-(u_{n-1}x_4 + u_n)}{u_{n-2}x_4 + u_{n-1}}.$$

Every plane through O is invariant and contains a perspective J involution from O. If u_{n-2} is identically zero, the fixed surface is itself a monoid, and P' is the harmonic conjugate of P as to O and the residual point of intersection with OP. The basis curve consists of the lines common to the two cones $u_{n-1} = 0$, $u_n = 0$, which are fundamental of the second kind, that is, the image of any point on any one of them is the whole line passing through it. The image of a point on a k-fold basis line is the whole line counted k-fold.

Of the other types, any one is expressible in the form

$$rx_i' = \varphi_i,\ i = 1, 2, 3; \quad x_4' = \frac{x_4 \varphi u + v}{w x_4 - u x},$$

in which the φ_i define an involutorial Cremona net of cones with common vertex at O, φ is a linear function of the φ_i and x is the same linear function of the x_i. The u, v, w are algebraic polynomials in functions k_i that are invariant under the ternary involution.

In particular, take BM. The φ_i are of order 17, and have eight basis lines each to order 6. Now suppose $n = 6$, and v, w are identically zero. The result is of order 18, the image of O is the plane $x = 0$ passing through O, and every line joining a pair of conjugate points meets the fundamental line m: $x = 0, x_4 = 0$. The conjugate of an arbitrary line is a rational curve of order 18, which passes simply through O. If the given line meets m, then the image of the point of intersection is a component of the conjugate; the proper image is a plane curve of order 17 in the plane of m and the given line. It is a plane B transformation. The line meets the invariant cone $R_9 = 0$ of B in 9 points, which also lie on the conjugate C_{17}. Hence the line meets C_{17} in four pairs of conjugates. In this particular monoidal transformation any line joining a pair of conjugate points belongs to a special linear complex and contains three other pairs of conjugate points. This probably includes irrational involutions.

In connection with the monoidal types of birational transformations we have an immediate example of the existence of (n, n^2) types, found synthetically by Tummarello(45) for every value of n, although earlier writers supposed they could not exist for $n > 3$(46).

The equations of the transformation are of the form

$$x_i' = \varphi_i, \ (i = 1, 2, 3), \quad x_4' = u_{n-1}x_4 + u_n,$$

in which $|\varphi|$ is a Cremona net of cones of order n with vertices at $(0, 0, 0, 1)$.

7. *Classification.* The first proposal was to classify involutions, like other birational transformations, according to the order of the surfaces conjugate to the planes of space. This was done for those of order 2 in the earliest

memoirs. It was soon found that those of order 3 could be expressed by means of three symmetric bilinear equations in the coordinates of two variable conjugate points(47). The types arising from three polarities have been established by Morris(48), but a number of interesting additional types appear when one or two of the polarities are replaced by linear complexes, either one or both of which may be general or special. These types offer no new difficulty, but no enumeration is complete without them.

But this method was early recognized to be impracticable, as no general criteria are known, to insure the involutorial character of the transformation when $n > 3$. A real advance was that of classifying involutions according to the genus of a general plane section of the conjugate surface of a plane(49). The first category is that of genus zero. The conjugate surface must be either a quadric, a Steiner quartic, or a ruled surface of order n with an $(n-1)$-fold line. Those of the first category were first obtained by Aschieri(50) and by Martinetti(42), those of the second by Montesano(51), and those of the third by Montesano(40). This method has not been developed further, on account of lack of knowledge of forms of rational surfaces with plane sections of given genus. Those of genus 1 have recently been made by Nobile(80).

8. *Line Complex containing the Involution.* The next procedure is that of considering the system of lines obtained by joining conjugate points. This system may be either a congruence or a complex. In the former case each line of the congruence contains an infinite number of pairs of conjugates. These congruences are always of order 1(35), and the transformations reducible to monoidal types. When the lines form a complex, each line contains one or more pairs of conjugate points. The linear complex, each line containing one pair, was considered by Montesano(52).

This does not give an extensive list of involutions, but each involution is determined by a particular curve of order 10 and genus 11. Given a hyperelliptic curve C_6^3 of order 6 and genus 3. Two quartic surfaces through it determine a C_{10}^{11} through which pass ∞^4 quartic surfaces. Of this system, the web passing through a point P also intersects in its conjugate P'. The lines PP' generate a linear complex. An arbitrary plane meets C_{10}^{11} in 10 points lying on a cubic curve. The curve may break up into a particular(53) C_7^5 and a cubic. The involution is of order 11, and may be particularized to orders $10 \cdots 3$. Types belonging to a special linear complex were later studied by Pieri(54), but not all such types were found. They can all be expressed by a single formula, given by Sharpe and me(17). All of the monoidal types JM are included in this category, as well as the simple M.

Involutions belonging to the tetrahedral complex were derived by Montesano(55), and later from another standpoint by Wimmer(56). This type occurs incidentally in the discussion of Sharpe and Snyder(17). The involution is of order 19. Each surface of a web of quartics having a basis C_{11}^{14} is left invariant by the involution. Other cases of involutions belonging to special quadratic complexes were studied by Pieri(57), and of involutions belonging to complexes of tangents by Pieri(58), and by Montesano(59). They can all be reduced to monoidal types. The question now arises whether an involution belongs to every quadratic complex, and the answer is No. (60) The study of the complex to which an involution belongs is of great service, but it does not furnish a satisfactory basis for classification. Montesano(61) showed that involutions in which each conic of a linear system is self conjugate belonged to the type studied by De Paolis(35), hence reducible to monoidal forms.

In all these cases, each line of the complex contains a single pair of conjugate points, but involutions exist in which each line of the complex contains two, three, or four pairs of conjugate forms(44). Montesano has proposed to me to classify involutions according to the minimum product of the order of the complex to which it belongs and the number of pairs of points on each line of the complex.

9. *Irrational Involutions.* It is known that irrational involutions in space exist. The first example was given by Enriques (62), and later simplified by Aprile (63). But both of these are of order higher than two, and it is not known whether irrational involutions of order 2 exist or not. But there are three that make it seem very probable. The first was discussed by Montesano (64). Consider the group of order 32, $x_i' = \pm x_i$ which leaves the equations $\sum x_i^2 = 0$, $\sum a_i x_i^2 = 0$ invariant. Think of the x_i as line-coordinates, and map the quadratic complex defined by them on the points of space. The central perspective transformations go into six perspective monoidal ones of order 3, all the invariant cubic surfaces passing through a common space quintic of genus 2. The product of two of the central perspectives corresponds to cubic involutions which leave every plane through the line joining the centers invariant, and every point of an elliptic space quartic remains fixed, the curve being the residual to the quintic, intersection of the invariant cubic surfaces of the monoidal components. The ten products by threes $(123 = 456)$ are more complicated. This depends upon the representation of the lines upon the point space, and the configuration is included in the next case, also considered by Montesano (37). The following outline is based upon a study being made by Professor Sharpe and me, not yet completed. Consider the involution I in which the conjugate of a plane is a surface of order n, having a line p to multiplicity $n-2$. There is also a basis curve of order $3n-4$ meeting p in $3n-7$ points. Planes through p are transformed into planes through p, either each into itself, or the pencil is in involution, with two invariant planes. If p is defined by $x_1 = 0$, $x_2 = 0$ we may write

$$x_1' = Mx_1, \quad x_2' = -Mx_2, \quad x_3' = E, \quad x_4' = F.$$

Since the transformation is birational and involutorial, it follows that x_3, x_4 are connected with x_3', x_4' by a bilinear equation. It is now easy to get the equation of the complex to which the involution belongs. It is always a particular form, being most nearly general for $n = 3$. It has been shown by Fano(65) that the general cubic complex is irrational. In some particular cases the spatial, surface, and arithmetic genera were all found to be zero, but these conditions to not insure the rationality of the involution, since a three-dimensional variety may have all its genera zero and still be irrational(65). The third illustration is furnished by the general cubic variety V in four space(67). Pass a plane through any point P on V and any line l lying on it. The plane section consists of l and a conic meeting it in two points, A, B. The lines AP, BP are tangent to V at points on l. Conversely, given any tangent to V at a point of l. It meets V in a residual point P, hence there is a $(1, 2)$ correspondence between the points of V, and its tangents at points of l. If the tangents are now mapped on the points of an auxiliary three space, we obtain an involution belonging to V(68). It is of order 6, and belongs to the type JM, hence also to a linear complex, but every line joining a pair of conjugate points contains one other pair. Thus it is mapped on a special linear complex doubly. But we have practically no criteria of rationality of varieties which can be mapped doubly on rational three spaces(66).

10. *Curves and Surfaces Invariant under Involutions.* In every rational involution every surface of a web remains invariant, and also the curves of intersection of pairs of surfaces of the web. In this way many properties of curves can be obtained which belong to given irrational involutions of any given genus. Associated with every curve of (x'), not a contact curve of L', is a curve of (x) which is invariant under the involution, the pairs of conjugate points belonging to the given curve in (x'). And we may have surfaces belonging to more than one web, each of which defines an involution. On this surface, the two involutions generate a group which may be finite or infinite. Numerous examples of such surfaces have been given(69), but many of them are

disposed of by the theorem of Enriques (70) that such a surface has a pencil of elliptic curves, except when $p_a = P_2 = 1$. In every case thus far found, all the operations can be expressed in terms of Cremona transformations. A surface having $p_a = P_2 = 1$ but not containing a pencil of elliptic curves was found by Fano (71), and its group discussed by Severi (72). That it belongs to two webs connected with (1, 2) correspondence was shown by Sharpe (32) and me. A second example was found by us (73), having similar properties, defined by the web of quartic surfaces through a space curve of order 8 and genus 2. A detailed discussion of all these types, with particular reference to their groups, was given by Fano (74).

But another quartic surface exists (73), for which $p_a = P_2 = 1$, which has no elliptic curves, and is invariant under an infinite discontinuous group, but not generated by involutions (75). The operations of this group may also be expressed by Cremona transformations. The surface is defined by containing a sextic curve of genus 3.

Problems.

1. What is the definition of a type of space involution?
2. Can rational involutions be classified in a finite number of types?
3. Do irrational involutions of order two exist?
4. Do all surfaces invariant under infinite discontinous groups belong to Cremona groups?

   For example, given a quartic surface with five nodes. The space cubic determined by them and a point P on the surface meets it again in P', defining an involution under which the surface is invariant. Can this be expressed as a Cremona transformation? Now take a quartic with six nodes. There are six such involutions. Do they generate an infinite group? These illustrations may be trivial, since the surface contains pencils of elliptic curves.
5. Can the equivalence of two given involutions be expressed in terms of rational invariants?

This is a short list, but will probably suffice for some time. Encouraged by recent contributions to the theory of postulation and equivalence, as given by Hilda Hudson(76) and by Severi, and illustrated by Tummarello and by Miss Moffa, and by the important theorem of Montesano(77) concerning multiple parasitic lines, let us continue to hope that a final answer may be found to some of them. The recent researches of Chisini(78) on the singularities of algebraic surfaces, and the proposal of Severi(79) concerning classification of space curves will be of fundamental assistance.

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