as H. J. S. Smith, and Vandiver's numerous contributions to the same end. Last there is the recent work of Dickson, which bids fair to be epoch making, on *Algebras and their Arithmetics*,* where the classic theory of algebraic numbers finds a simple and profound generalization.

With the books of Landau and Dickson, the report of Hilbert and that of the present authors now available, it is to be hoped that algebraic numbers, one of the major divisions of modern mathematics, will not much longer remain in learned obscurity, but will take its rightful place as one of the chief glories of any liberal mathematical education.

E. T. BELL

VOLUMES II AND III OF DICKSON'S HISTORY

History of the Theory of Numbers. By L. E. Dickson. Vol. II, *Diophantine Analysis,* xxv + 803 pp. Vol. III, *Quadratic and Higher Forms,* (with a chapter on the class number by G. H. Cresse). iv + 313 pp. The Carnegie Institution, Washington, 1923.

Since the time of Gauss, the theory of numbers has developed in a number of different directions. Let us examine this development prior to the year 1890. Dirichlet and Riemann founded the analytic prime number theory; Kummer, Kronecker, and Dedekind created the theory of algebraic numbers; Eisenstein, Hermite, Smith, and Minkowski developed the arithmetic theory of forms; Jacobi, Eisenstein, Kronecker, Smith, and Hermite applied the theory of elliptic functions to various problems. It will be noted that, in the main, this progress centered about a few great names. The discoveries of these men did not excite the attention of other mathematicians in many cases because the contents of the original papers were often complicated and difficult to read, and few suitable texts were provided to meet the needs of the beginner.

In considering the period between 1890 and 1900, however, a decided change is noted. In this interval appeared the *Lehrbuch der Algebra* of Weber and Hilbert's *Bericht über die Theorie der Algebraischen Zahlen.* These works and the original papers of the same authors appear to have exercised a profound influence on a number of able young mathematicians. In another line, Hadamand and de la Vallée Poussin obtained epoch making results in the theory of the Riemann zeta function, with applications to the asymptotic distribution of prime numbers. Minkowski founded a geometry of numbers which has bearing on many parts of the number theory. Dickson initiated his extensive contributions to the subject by developing the theory of finite fields,

* This sentence was written by the reviewer before the award to Professor Dickson of the Cincinnati Prize. See page 90 of this issue. THE EDITORS.

having direct connection with the theory of congruences, and attacked arithmetic problems arising in the theory of linear groups.

Since 1900 the theory of numbers has been investigated not only by a large proportion of mathematicians of the first rank, but by a great number of less gifted individuals, until at the present time it would perhaps be unwise to assert that the subject has not received sufficient attention from men of research.

The progress has been continuous, and, it seems to the writer, comparable with that during any similar interval of time in the past. In particular, it does not appear unlikely that the year 1909 will be referred to in the future as a landmark in the theory of numbers. In that year A. Thüe published his proof of the theorem that the equation

$$f(x, y) = c,$$

where $f(x, y)$ is a homogeneous binary form of degree n with integral coefficients, irreducible in the rational field, and c is an integer, has only a finite number of solutions if $n > 2$; Hilbert's proof of Waring's theorem that any integer can be expressed as the sum of not more than k nth powers, k being an integer depending on n only, appeared; Wieferich showed that if

$$x^p + y^p + z^p = 0$$

has a solution in integers, x, y, and z prime to the odd prime p, then

$$2^{p-1} \equiv 1 \pmod{p^2};$$

and Furtwängler announced and proved a general theorem of reciprocity between two integers in a relative cyclotomic field of which the law of quadratic reciprocity between two odd rational integers is a special case. Each of these results attracted much attention and was the starting point of investigations of wide scope.

In recent years the subject has developed rapidly and in various ways. Based on important discoveries by Hecke, Landau and others regarding the Dedekind function, the methods employed in treating the theory of the distribution of rational primes have been extended in such a way as to yield corresponding theorems concerning the distribution of ideal primes in an algebraic field. Furtwängler and Fueter developed the theory of the class field (Klassenkörper) of an algebraic field, which has close connection with the higher laws of reciprocity and the complex multiplication of elliptic functions. Dickson has been the leader in applications of modern algebraic methods to various problems; has founded theories of modular invariants, and of arithmetics of linear associative algebras, the latter having important uses in ordinary Diophantine analysis. Hardy and Littlewood have discovered new analytic methods in additive number theory, obtaining therefrom remarkable results concerning partitions, representations of a number as sums of squares, Waring's theorem (referred to above) and the representation of an integer as the sum of primes. Humbert, Mordell, and others have contributed extensively in recent years to the arithmetic theory of forms.

It is fitting, during such an interesting stage in the development of number theory, that the monumental work under review should appear. Volume I of the *History* was reviewed by D. N. Lehmer in this BULLETIN (vol. 26 (1919), p. 125).

The plan of the history is to cover all parts of the subject with the exception of algebraic numbers. Volume IV, which has not as yet appeared, is to comprehend, in particular, the extensive literature on the laws of quadratic reciprocity.

For the second volume Dickson has given a resumé of the contents of each chapter in the preface; in the third volume a resumé is at the beginning of each chapter. In view of this we shall not take up in detail the material covered, aside from a few points which expecially interest the reviewer.

The chapter headings of volume II are: Polygonal, pyramidal and figurate numbers; Linear Diophantine equations and congruences; Partitions; Rational right triangles; Triangles, Quadrilaterals and tetrahedra; Sum of two squares; Sum of three squares; Sum of four squares; Sum of n squares; Number of solutions of quadratic congruences in n unknowns; Liouville's series of eighteen articles; Pell equation; Further single equations of the second degree; Squares in arithmetic or geometric progression; Two or more linear functions made squares; Two quadratic functions of one or two unknowns made squares; Systems of two equations of degree two; Three or more quadratic functions of one or two unknowns made squares; Systems of three or more equations of degree two in three or more unknowns, Quadratic form made an nth power; Equations of degree three; Equations of degree four; Equations of degree n; Sets of integers with equal sums of like powers; Waring's problem and related results; Fermat's last theorem,

$$ax^r + by^s = cz^t,$$

and the congruence

$$x^n + y^n \equiv z^n \pmod{p}.$$

Volume III contains: Reduction and equivalence of binary quadratic forms; Representation of integers; Explicit values of x, y in

$$x^2 + \Delta y^2 = g;$$

Composition of binary quadratic forms; Orders and genera and their composition; Irregular determinants; Number of classes of binary quadratic forms with integral coefficients; Binary quadratic forms whose coefficients are complex integers or integers of a field; Number of classes of binary quadratic forms with complex integral coefficients; Ternary quadratic forms; Quaternary quadratic forms; Quadratic forms in n variables; Binary cubic forms; Cubic forms in 3 or more variables; Forms of degree $n \geqq 4$; Binary Hermitian forms; Hermitian forms in n variables and their conjugates; Bilinear forms, matrices, linear substitutions; Representation by polynomials modulo p; Congruencial theory of forms.

On reading the chapter entitled "Pell's Equation," which cites 314 references, one is impressed by the number of writers who independently obtained special methods of solution approximating in character the well known continued fraction algorithm. This seems particularly noteworthy as an example of the evolution of a mathematical idea.

Those chapters in volume II beginning with "Squares in arithmetic or geometric progression" and ending with "Equations of degree n," seem to the writer to be a very unusual contribution to mathematical literature. They deal with subjects which particularly interest amateur investigators and a large proportion of the references are to be found in the problem columns of lesser known publications. Nothing has ever appeared which compares to these chapters in furnishing information about special Diophantine problems.

It will be seen from the above list that the material treated in the third volume is much more technical, in the main, than that taken up in the other volumes and the form in which some of the chapters have been written indicates that more space has been devoted to explanations of various general theories. The chapter on the binary quadratic form class number contains 375 references, indicating the fascination which this topic has had for numerous writers. Legendre, Gauss, Jacobi, Cauchy, Dirichlet, Eisenstein, Smith, Dedekind, Liouville, Hermite, Kronecker, Poincaré, Weber and Klein are some of the well known names associated with it.

Perhaps less known than the material in the other chapters is that contained in the chapters on cubic and Hermitian forms. Following Eisenstein, consider the cubic form

$$f = (a, b, c, d) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

with rational integral coefficients, and set

$$F = Ax^2 + Bxy + Cy^2,\ A = b^2 - ac,\ B = bc - ad,\ C = c^2 - bd.$$

The *determinant* of the quadratic form for $2F$ is called the determinant D of the cubic form f.

Using linear transformation, the terms proper, improper, equivalent, and class are defined for f in a manner analogous to that in quadratic form theory. Also, correspondences are obtained between the classes of F and certain classes of f. The representation of an integer by f depends on finding integers representable by F, (if $(A, B, C) = 1$), whose cubes can be represented in the form

$$V^2 - DV^2.$$

(Cf. the *History*, Chap. XII, Eisenstein[2], Pepin[13]). The topic has been investigated extensively by Eisenstein, Arndt, and Pepin.

After Hermite we write

$$v = x + iy, \; v_0 = x - iy,$$

and u_0 for the conjugate imaginary of u, and we have the binary Hermitian form

$$f(v, u;\; v_0, u_0) = Avv_0 + Bvu_0 + B_0 v_0 u + Cuu_0$$

where A and C are real, while B, B_0 are conjugate imaginaries. Then f takes only real values under a linear transformation and $BB_0 - AC$ is an invariant, called the determinant of f. Definite, indefinite, and reduced forms are defined and a theory has been built up which is analogous to that of binary quadratic forms. The subject has applications to the theory of quaternary quadratic forms and has been developed mainly by Hermite, Picard, Bianchi and Humbert.

We shall now consider the place of the work in mathematical literature. On page xx of the preface of volume II the author says:

"Conventional histories take for granted that each fact has been discovered by a natural series of deductions from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties. What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history." In the reviewer's opinion, Dickson has come very close to realizing these ideals, and this circumstance gives the work its greatest value. In particular, it appears that he has had always in mind the needs of the investigator. An endeavor is made to supply every scrap of information that might possibly aid in research on a particular problem.

It is characteristic of number theory that from time to time results which are quite new and of an entirely elementary character are obtained by methods also elementary. This being the case, one often finds it of value to study papers as old as those of Euler, Lagrange and Legendre, as *all* of their ideas have not yet been incorporated in general theories, but are, however, reported on in full in the *History*.

Muir's *History of the Theory of Determinants*, which resembles Dickson's *History* closely in giving detailed reports of each paper cited, differs, however, in other respects. Much space is devoted by Muir to criticising the results of articles or in praise of various writers. This type of material does not, in general, particularly interest the man of research, and the published opinions of individuals as to the value of mathematical results rarely have any interest after a term of years. This brings out a quality of Dickson's *Hystory* which it seems well to emphasize, namely, that these volumes may be regarded as a permanent foundation on which to bring the history of number theory up to date at various intervals in the future. It is hardly conceivable that anyone should want to re-examine all the material that Dickson has already gone over and re-write his work. He has put things in such form that no more is perhaps needed in order to cover future progress in number theory than to make additions to his work from time to time.

The material shows in itself how useful the history will be in enabling investigators to avoid duplication in published results. It is to be noted in glancing over the pages, how often the contents of a published article, is, unknown to the writer of it, largely a repetition of previous research of some other author.

After some experience in the use of the first and second volumes, the reviewer has noted a number of instances, where the author has expressed in a nut-shell the main results of a long and involved paper in a much clearer way than the writer of the article did himself. The ability to reduce complicated mathematical arguments to simple and elementary terms is highly developed in Dickson.

It often happens in the history of mathematics that a mathematician becomes a specialist in a particular topic, and, after years of experience with it, he publishes a treatise giving a harmonious and comprehensive development of the subject, the material being all arranged and presented according to his own particular point of view. This treatise may become a classic and its readers are likely to get in the habit of ignoring, to a considerable extent, the literature that preceded its publication. In this way the points of view of the older writers are often lost sight of, as these treatises rarely, if ever, reproduce all the older material on a particular topic. It would seem that there is too great a preponderance of books of this sort in the literature and too few histories or reports of the type of Dickson's work.

It may happen that some readers of this review are not interested particularly in number theory, but are specialists in some other field. Let them consider what a work of similar character to Dickson's *History* on their favorite subject would mean to them. Perhaps they will then be convinced of the supreme importance of producing detailed histories or reports in all branches of mathematics.

H. S. VANDIVER