A QUALITATIVE DEFINITION OF THE POTENTIAL FUNCTIONS*

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1. *Introduction.* In this paper we aim to set up postulates completely characterizing the potential functions, which do not involve derivatives or integrals, and are thus of a more qualitative nature than the definitions previously given. Incidentally, we may interpret all of our postulates as statements of properties of such physical quantities as give rise to potential functions, and when so interpreted, we see from physical grounds that they hold for the quantities in question. They thus furnish a means of going directly from certain physical problems to the potential functions, without the use of Laplace's equation. Our results will be stated in full only for potential functions of two and three variables, although they may evidently be extended to the n-dimensional case.

2. *Postulates for Two Dimensions.* Consider a class of functions of two variables, x and y, thought of for convenience as Cartesian coordinates, and let each function have associated with it a region R of the plane. Our assumptions are:

(1) Each function is continuous in both variables at all interior points of its region R.

(2) If R_1 and R_2, the regions for two functions of the class $f_1(x, y)$ and $f_2(x, y)$, have a region R_3 in common, then any linear combination of these functions, such as $Af_1(x, y) + Bf_2(x, y)$, is a function of the class, whose region contains all the points of R_3.

(3) If an orthogonal transformation of the variables (i. e., a change of Cartesian axes) converts the function $f(x, y)$ with region R into $F(x', y')$, this latter function is a member of the class, its region being the region R expressed in the new variables.

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(4) The function $f(x, y) = 1$ is a function of the class, its region R being the entire plane.

(5) If a given circle lies entirely inside each region R_i, belonging to $f_i(x, y)$, these $f_i(x, y)$ constituting an infinite sequence of functions of the class, and if, for points x_c, y_c on the circumference of the circle $\lim_{i \to \infty} f_i(x_c, y_c) = 0$, the limit being approached uniformly over the circumference, then the sequence of values of the functions $f_i(x, y)$ at the center of the circle cannot approach a limit different from zero.

We shall show that any family of functions satisfying these five postulates is necessarily a class of potential functions, each function being harmonic at any interior point of its region R. Since every linear class of harmonic functions satisfies the above postulates, the widest class of functions satisfying them is the totality of harmonic functions.

By way of motivating our choice of postulates, we note that postulates 1, 2, and 4 are natural requirements since the functions we are trying to define satisfy a linear homogeneous partial differential equation. Postulate 3 is allied to the fact that the Laplacian operator is the only linear differential operator of the second order invariant under orthogonal transformations of coordinates. The last postulate is added as a fairly weak condition which completes the characterization, since the preceding ones are not by themselves sufficient for this.

It is illuminating to consider the significance of the postulates for a physical example, say the functions giving the temperature of points of a set of thin plates, when various boundary temperatures are assigned. The postulates state, essentially, that these functions are continuous point functions, as a class independent of the choice of coordinates, which possess the property of combination by superposition; that if the boundary temperatures are constant, the temperature inside will be this same constant, and finally that if the temperatures at the circumference of a circular plate are altered so that they approach zero uniformly, the temperature at the center of the circular plate will approach zero.

3. *Deduction of the Mean-Value Property.* We shall now prove that the class of functions which satisfy the postulates of § 2 have the mean-value property, that is, if x_0, y_0 is the center of a circle C lying entirely inside of R, the region for $f(x, y)$, any function of the class, then $f(x_0, y_0)$, its value at the center of C is equal to the average value of $f(x, y)$ taken over the circumference of C.

To show this, we first form a series of positive constants ε_i such that $\lim_{i\to\infty} \varepsilon_i = 0$. We select a particular ε_i, and divide the circumference of the circle C into N_i equal parts, taking N_i so large that the oscillation of $f(x, y)$ on any one part is less than ε_i. This is possible since postulate 1 makes $f(x,y)$ continuous, and therefore uniformly continuous on the circumference of C. We next form $N_i - 1$ new functions, existing in and on C, obtained from $f(x, y)$ by rotating the axes about the center of C through an angle $2k\pi/N_i$ $(0 < k < N_i)$. These functions, by postulate 3, are members of the class, as is also the function $f_i(x, y)$, defined as the sum of these $N_i - 1$ functions and the original function, divided by N_i, by postulate 2. Finally, from postulates 2 and 4, we see that the function $F_i(x, y) = f_i(x, y) - f_c$, where f_c denotes the average value of the original function on the boundary of C, is also in the class.

The function $F_i(x, y)$ satisfies the inequality

$$(1) \qquad |F_i(x_c, y_c)| < \varepsilon_i,$$

where x_c, y_c denote any point on C. For, if p_k is the average value of $f(x, y)$ in the kth subdivision of C, we have

$$(2) \qquad f_c = \sum_{k=1}^{N_i} p_k / N_i,$$

and from the choice of N_i, for x_c, y_c in the kth subdivision

$$(3) \qquad |f(x_c, y_c) - p_k| < \varepsilon_i.$$

Recalling the method of forming $f_i(x, y)$, we deduce from (2) and (3) that

$$(4) \qquad |f_i(x_c, y_c) - f_c| < \varepsilon_i,$$

for all points on the circumference of C, and this is equivalent to (1) above.

Moreover, at x_0, y_0, the center of the circle C, we have

(5) $$F_i(x_0, y_0) = f_i(x_0, y_0) - f_c = f(x_0, y_0) - f_c.$$

Thus the values of the functions $F_i(x, y)$ at x_0, y_0 are the same for all values of i, and consequently approach a limit when i becomes infinite. But from (1) and postulate 5, this limit must be zero, and (5) gives

(6) $$f(x_0, y_0) = f_c,$$

which is the mean-value property.

4. *The Harmonic Character of the Functions.* We may now readily prove that the functions of our class are harmonic at all interior points of their regions R. For, with any interior point of a region R as center, let us draw a circle lying wholly inside R, the region for the function $f(x, y)$. We also form, by Poisson's integral or otherwise, a harmonic function $h(x, y)$ having the same values along the circumference of the circle as the given function $f(x, y)$. Since both $h(x, y)$ and $f(x, y)$ possess the mean-value property, their difference $f(x, y) - h(x, y)$ will also possess it. Consequently this function takes on its maximum and minimum values on the circumference of the circle; and since it is zero on the circle, it must be zero identically. Hence $f(x, y)$ agrees with $h(x, y)$ in a neighborhood of the center of the circle and is harmonic at this point.

5. *Postulates for Three Dimensions.* It is fairly obvious what revisions we must make in the postulates of § 2 to make them applicable to potential functions in three-dimensional space. Here we deal with a class of functions of three variables, or Cartesian coordinates, x, y and z, where each function has associated with it a three-dimensional region R, of space. Our assumptions now are:

(1) Each function is continuous in all three variables at all interior points of its region R.

(2) If R_1 and R_2, the regions for two functions of the class $f_1(x, y, z)$ and $f_2(x, y, z)$, have a region R_3 in common, then any linear combination of these functions, such as $Af_1(x, y, z) + Bf_2(x, y, z)$, is a function of the class, whose region contains all the points of R_3.

(3) If an orthogonal transformation of the variables (i. e., a change of Cartesian axes) converts the function $f(x, y, z)$ with region R into $F(x', y', z')$, this latter function is a member of the class, its region being the region R expressed in the new variables.

(4) The function $f(x, y) = 1$ is a function of the class, its region R being the whole of space.

(5) If a given sphere lies entirely inside each region R_i, belonging to $f_i(x, y, z)$, these $f_i(x, y, z)$ constituting an infinite sequence of functions of the class, and if, for points x_s, y_s, z_s on the surface of the sphere $\lim_{i\to\infty} f_i(x_s, y_s, z_s) = 0$, the limit being approached uniformly over the surface; then the sequence of values of the functions $f_i(x, y, z)$ at the center of the sphere cannot approach a limit different from zero.

Any class of functions satisfying these five postulates is, as we shall show presently, a class of Newtonian potential functions. Consequently the widest class of functions satisfying them is the totality of Newtonian potential functions. The remarks made about our earlier set of postulates at the end of § 2 apply here, mutatis mutandis.

Our reason for writing out the postulates at length and carrying out the proof of their sufficiency for the three-dimensional case, is that the transition to the three-dimensional case requires an essential modification in the proof. For, in the earlier proof of § 3, use was made of a series of regular polygons approximating a circle. As no analogous configuration exists in space, we must resort to a property of functions on a sphere, to which we next proceed.

6. *Functions on a Sphere.* If we are given a function on a sphere, we may form a new function from it by the following process. We select a particular diameter of the sphere as an *axis*, and so determine the second function that its value is constant on every circle on the sphere whose plane is perpendicular to the axis, and is equal to the average of the original function on this same circle. We define this process as that of *averaging the original*

function about an axis. We may now state the following lemma.

LEMMA. *If a continuous function on a sphere is averaged about one axis, the resulting function about a second axis perpendicular to the first one, this third function about the first axis, and so on, indefinitely, using the two axes alternately, the sequence of functions so obtained will approach a limit. Furthermore, this limit will be constant over the whole sphere, and equal to the average of the original function over the sphere.*

To prove this, we first observe from the continuity of the function that it possesses a maximum and a minimum value, which are actually reached, since the surface of the sphere is closed. Also, from the nature of the averaging process, the successive functions are all continuous, and the maximum values M_i form a never increasing sequence, while the minimum values m_i form a never decreasing one. Hence these quantities approach limits, which we denote by M and m respectively.

Another consequence of the character of the averaging process is that if we determine a positive δ such that the oscillation of our original function is less than a preassigned ε in every circle on the sphere of radius δ, which we may do since this function is continuous and therefore uniformly continuous on the sphere, the same δ will retain its relation to ε after the averaging, and hence for all subsequent functions in the series. This follows from the fact that the oscillation in a circle of radius δ after averaging cannot exceed the maximum oscillation in the set of circles of equal radius having their centers in the plane perpendicular to the axis containing the center of the original circle.

Let us now select an ε and determine a δ in the way described above, and let us set

$$\eta = \frac{2\varepsilon\delta}{\pi r}, \tag{7}$$

where r is the radius of the sphere. We next consider the result of averaging the original function i times, taking i so great that F_i, the resulting function, has its maximum and minimum values each within η of their respective limits, i. e.,

$$(8)\qquad \begin{cases} M \leqq M_i \leqq M + \eta, \\ m - \eta \leqq m_i \leqq m. \end{cases}$$

Let E be the constant value of the function F_i on the equator, i. e., on the great circle perpendicular to the axis about which we have just averaged, and consequently having as one diameter the other axis which we must use to get F_{i+1}. From the way we selected δ, F_i cannot differ from E by more than ε in the zone of width 2δ bounded by small circles parallel to and at distances δ above and below the equator. Hence, if we construct two lunes lying entirely in this zone, each of angular width $2\delta/r$, bounded by planes through the new axis, the value of F_i will be at most $E + \varepsilon$ inside these lunes. The remaining portion of the sphere consists of two lunes each of angular width $\pi - 2\delta/r$, and in these lunes, the function is at most M_i or $M + \eta$ by (8). Consequently, if we average once more to get F_{i+1}, since this function cannot exceed the average of the upper bounds in the lunes in question, we must have

$$(9)\qquad F_{i+1} \leqq \frac{\frac{4\delta}{r}(E+\varepsilon) + \left(2\pi - \frac{4\delta}{r}\right)(M+\eta)}{2\pi}.$$

Since F_{i+1} is at least as great as M at some point, this implies that

$$(10)\qquad M \leqq M + \eta + \frac{2\delta}{\pi r}(E + \varepsilon - M - \eta),$$

from which, using (7), we get

$$(11)\qquad E \geqq M + \eta - 2\varepsilon \geqq M - 2\varepsilon.$$

By an entirely analogous argument, using the minimum values of F_i and F_{i+1}, we show that

$$(12)\qquad E \leqq m + 2\varepsilon.$$

These last two equations give

$$(13)\qquad E + 2\varepsilon \geqq M \geqq m \geqq E - 2\varepsilon,$$

which shows that $M = m$, since ε can be made arbitrarily small.

Hence the value of the function at all points approaches this constant value, and since the averaging process leaves the average of the function taken over the entire sphere unchanged, the constant must be equal to this average, or the average of the original function over the sphere.

7. *Deduction of the Mean-value Property.* We are now in a position to prove that the class of functions which satisfy the postulates of § 5 have the mean-value property. That is, if x_0, y_0, z_0 is the center of a sphere S lying entirely inside of R, the region for $f(x, y, z)$ any function of the class, then $f(x_0, y_0, z_0)$, its value at the center of S, is equal to the average value of $f(x, y, z)$ taken over the surface of S.

Since the process of averaging about an axis discussed in the preceding section involves limiting processes not covered by our postulates, we cannot directly apply it; but we can obtain a function of our class from $f(x, y, z)$ having the same value at the center of S, and on S approximating the result of averaging $f(x, y, z)$ about any axis as closely as we please. For we have merely to select an N so large that the oscillation of the original function in any circle of radius $\pi r/N$ is less than ε, form functions obtained from the original function $f(x, y, z)$ by rotating the coordinates about the chosen axis through angles $2k\pi/N$ $(0 < k < N)$, sum these functions and divide by N. The proof that this is the desired function is entirely parallel to the argument given in § 3.

Let us now form a series of positive constants ε_i such that $\lim_{i \to \infty} \varepsilon_i = 0$. For each ε_i we shall form a function $f_i(x, y, z)$ as follows. First form a function $f_{i1}(x, y, z)$ which has the same value at the center of S as $f(x, y, z)$ and which, on the surface of S, differs from the result of averaging $f(x, y, z)$ about a fixed axis by less than $\varepsilon_i/4$, which can be done by the method just described. Next select a second axis perpendicular to the first, and form $f_{i2}(x, y, z)$ which, on the surface of S, differs from the result of averaging $f_{i1}(x, y, z)$ about this second axis by less than $\varepsilon_i/8$. Then form $f_{i3}(x, y, z)$ differing on the surface of S from the result of averaging $f_{i2}(x, y, z)$ about the first axis by less than $\varepsilon_i/16$ and so on; $f_{it}(x, y, z)$ differing on the surface of S from the result of averaging $f_{i,t-1}(x, y, z)$ about the first or second axis, according as t is odd or even, by less than $\varepsilon_i/2^{t+1}$. The functions on the surface of S, $F_1(x, y, z), \ldots, F_t(x, y, z), \ldots$ obtained from $f(x, y, z)$ by actually averaging about the two axes in succession, will be related to the set just constructed, in that, on the surface of S,

$$(14) \qquad |F_1 - f_{i1}| < \varepsilon_i/4, \qquad |F_2 - f_{i2}| < \varepsilon_i/2^2 + \varepsilon_i/2^3,$$

and for all values of t,

$$(15) \qquad |F_t - f_{i1}| < \varepsilon_i(1/2^2 + 1/2^3 + \cdots + 1/2^{t+1}) < \varepsilon_i/2.$$

But, by the lemma of § 6, the limit of $F_t(x, y, z)$ as t becomes infinite is f_s, the average of $f(x, y, z)$ over the surface of S. Hence, by taking t large enough, we may make

$$(16) \qquad |F_t - f_s| < \varepsilon_i/2; \qquad t \geqq T.$$

We put

$$(17) \qquad f_i(x, y, z) = f_{iT}(x, y, z).$$

Then, from (15) and (16), we have

$$(18) \qquad |f_i(x, y, z) - f_s| < \varepsilon_i.$$

Furthermore, $f_i(x_0, y_0, z_0)$ is equal to $f(x_0, y_0, z_0)$, since all the processes used to obtain $f_i(x, y, z)$ from $f(x, y, z)$ left its value at the center of S unchanged. Thus the sequence of functions

$$(19) \qquad G_i(x, y, z) = f_i(x, y, z) - f_s$$

satisfies all the conditions of our fifth postulate. Since its value at the center of S is constant, this constant must be zero, and we have

$$(20) \qquad f(x_0, y_0, z_0) = f_i(x_0, y_0, z_0) = f_s$$

which is the mean-value property.

8. *The Harmonic Character of the Functions.* We may now apply the reasoning of § 4 to show that inside any sphere S entirely within its region R, any function of our

class $f(x, y, z)$ is identical with the harmonic function set up by Poisson's integral which has the same values on the boundary of S as $f(x, y, z)$. Thus all the functions of our class are harmonic at all points interior to their regions. Also since every linear class of harmonic functions in three variables satisfies them, the widest such class of functions is the totality of Newtonian potential functions.

9. *Concluding Remarks.* We might extend our postulates to the case of n dimensions, the proof being accomplished by a proper generalization of the lemma of § 6.

It should also be noticed that the postulates for two dimensions apply to functions on a sphere* (and for three dimensions to functions on a hypersphere) provided we replace the orthogonal transformations which correspond to a rotation of the plane (or 3-space) by the transformations on the sphere (hypersphere) corresponding to rotations about a diameter, as the proof given still applies with but slight modifications.

Moreover, since postulate 5 is a consequence of the maximum-minimum property, which is, of course, much more restrictive, we may obtain a set of postulates, by postulating the maximum-minimum property, which, though more stringent than ours, is stated in terms of more familiar properties. Since this substitution enables us to dispense with postulate 4, the new assumptions may be stated in the following form:

A linear family of functions of two (three) variables, invariant as a family under orthogonal transformations of the variables, each function being continuous at all interior points of some region R, and such that for any subregion of R it takes its maximum and minimum values on the boundary of this region, is necessarily a family of harmonic functions.

This statement is of interest, because it shows one set of conditions which, on being added to the maximum-minimum property, characterize the class as harmonic.

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* This extension was called to my attention by Dr. Norbert Wiener.