COMPLETE SETS OF REPRESENTATIONS OF TWO-ELEMENT ALGEBRAS*

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1. *Introduction.* Any algebra consists of one or more classes of elements and one or more operations or relations among the elements. An operation $\oplus$ of a class K of two elements a, b is given by a $\oplus$-table of the form

$$\begin{array}{c|cc} \oplus & a & b \\ \hline a & c_1 & c_2 \\ b & c_3 & c_4 \end{array}$$

where the c's all belong to K, or do not all belong to K, according as the operation is K-closing or not. A relation R in K is given by an R-table of the form

$$\begin{array}{c|cc} R & a & b \\ \hline a & \pm & \pm \\ b & \pm & \pm \end{array}$$

where the sign $+$ indicates that aRb holds, the sign $-$ that aRb does not hold. The object of this paper is to give convenient representations of these $\oplus$-tables and R-tables, hence of all two-element algebras, and to point out the usefulness of these representations in connection with fundamental questions relating to postulate-sets.

2. *Representation of Class-closing Operations.* There are 2^4 operations $\oplus$ possible in a class of two elements when the condition of closure is satisfied. The following theorem gives us two sets of representations of these operations, one arithmetic and one boolean. In the boolean representation the symbols 0, 1, a', $a+b$, ab denote respectively

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(1) *An arithmetic representation of boolean logic,* (2) *Arithmetic independence systems for the Whitehead-Huntington postulates for boolean algebras,* (3) *A boolean representation of a number field.*

the logical "zero," the "whole," the negative of a, the logical sum of a and b, the logical product of a and b.

THEOREM A. *The* 2^4 *systems* $(K, \oplus)$ $(K: 0, 1;\ a \oplus b = 0, 1)$ *are equivalent to the arithmetic functions*

$$\text{(I)} \qquad A_1ab + A_2a + A_3b + A_4 \text{ modulo } 2, \qquad (A_i = 0, 1)$$

and also to the boolean functions

$$\text{(II)} \qquad Aab + Bab' + Ca'b + Da'b', \quad (A, B, C, D = 0, 1).$$

The theorem is true for the following reasons:

(1) Every system $(K, \oplus)$ of the theorem is equivalent to a $\oplus$-table of the form

$$\text{(III)} \qquad \begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & e_1 & e_2 \\ 1 & e_3 & e_4 \end{array} \qquad (e_i = 0, 1).$$

(2) Every function of form (I) or of form (II) determines a $\oplus$-table (III).

(3) Every $\oplus$-table (III) determines a function $f(a, b)$ of form (I) from the equations (modulo 2)

$$f(0, 0) = A_4, \qquad f(0, 1) = A_3 + A_4,$$
$$f(1, 0) = A_2 + A_4, \qquad f(1, 1) = A_1 + A_2 + A_3 + A_4.$$

(4) Every $\oplus$-table (III) determines a function $F(a, b)$ of form (II) from the fact that

$$F(0, 0) = D, \quad F(0, 1) = C, \quad F(1, 0) = B, \quad F(1, 1) = A.$$

Table A gives the representations determined by Theorem A.

3. *Representation of Relations.* Since functions (I), as well as functions (II), of Theorem A are all distinct, we have the following theorem.

THEOREM B. *The* 2^4 *systems* (K, R) $(K: 0, 1;$ *R a dyadic relation*) *are equivalent to the arithmetic equations* (*modulo* 2)

$$\text{(I}'\text{)} \qquad A_1ab + A_2a + A_3b + A_4 = 0, \qquad (A_i = 0, 1),$$

and also to the boolean equations

$$\text{(II}'\text{)} \quad Aab + Bab' + Ca'b + Da'b' = 0, \quad (A, B, C, D = 0, 1).$$

Table B gives the representations determined by Theorem B.

TABLE A

REPRESENTATIONS OF SYSTEMS $(K, \oplus)$

$(K\colon 0, 1;\ a \oplus b = 0, 1)$

No.	$a \oplus b$	Arithmetic Representation	Boolean Representation
1	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}$	0	0
2	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$	$ab \quad (\mathrm{mod}\ 2)$	ab
3	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}$	$ab+b \quad (\mathrm{mod}\ 2)$	$a'b$
4	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}$	b	$ab+a'b=b$
5	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}$	$ab+a \quad (\mathrm{mod}\ 2)$	ab'
6	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}$	a	$ab+ab'=a$
7	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$	$a+b \quad (\mathrm{mod}\ 2)$	$ab'+a'b$
8	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$	$ab+a+b \quad (\mathrm{mod}\ 2)$	$ab+ab'+a'b=a+b$
9	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}$	$ab+a+b+1 \quad (\mathrm{mod}\ 2)$	$a'b'$
10	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$	$a+b+1 \quad (\mathrm{mod}\ 2)$	$ab+a'b'$
11	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 0 \end{array}$	$a+1 \quad (\mathrm{mod}\ 2)$	$a'b+a'b'=a'$
12	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$	$ab+a+1 \quad (\mathrm{mod}\ 2)$	$ab+a'b+a'b'=(ab')'$
13	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}$	$b+1 \quad (\mathrm{mod}\ 2)$	$ab'+a'b'=b'$
14	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}$	$ab+b+1 \quad (\mathrm{mod}\ 2)$	$ab+ab'+a'b'=(a'b)'$
15	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}$	$ab+1 \quad (\mathrm{mod}\ 2)$	$ab'+a'b+a'b'=(ab)'=a'+b'$
16	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$	1	1

TABLE B

Representations of Systems (K, R)
(K : 0, 1; R a dyadic relation)

No.	$a\,R\,b$	Arithmetic Representation	Boolean Representation
1	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & - & - \end{array}$	$1=0$	$1=0$
2	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & - & + \end{array}$	$ab+1=0 \pmod 2$	$ab'+a'b+a'b' = (ab)' = a'+b'=0$
3	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & - \end{array}$	$ab+b+1=0 \pmod 2$	$ab+ab'+a'b' = (a'b)'=0$
4	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & + \end{array}$	$b+1=0 \pmod 2$	$ab'+a'b' = b' = 0$
5	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & - \end{array}$	$ab+a+1=0 \pmod 2$	$ab+a'b+a'b' = (ab')'=0$
6	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & + \end{array}$	$a+1=0 \pmod 2$	$a'b+a'b' = a' = 0$
7	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & + & - \end{array}$	$a+b+1=0 \pmod 2$	$ab+a'b'=0$
8	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & + & + \end{array}$	$ab+a+b+1=0 \pmod 2$	$a'b'=0$
9	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & - & - \end{array}$	$ab+a+b=0 \pmod 2$	$ab+ab'+a'b = a+b = 0$
10	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & - & + \end{array}$	$a+b=0 \pmod 2$	$ab'+a'b=0$
11	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & - & - \end{array}$	$a=0$	$ab+ab' = a = 0$
12	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & - & + \end{array}$	$ab+a=0 \pmod 2$	$ab'=0$
13	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & + & - \end{array}$	$b=0$	$ab+a'b = b = 0$
14	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & + & + \end{array}$	$ab+b=0 \pmod 2$	$a'b=0$
15	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & + & - \end{array}$	$ab=0 \pmod 2$	$ab=0$
16	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & + & + \end{array}$	$0=0$	$0=0$

4. *Representation of Operations that are not Class-closing.* Representations of operations $\oplus$ that do not satisfy the condition of closure are given by the following theorem.

THEOREM *C. Let* $\sum \equiv (K, \oplus)$ *be a system of two elements* 0, 1, *with* $\oplus$ *such that* $a \oplus b = x$* *not in* K *for some values of* a, b; *let* $f(a, b)$ *be an arithmetic function equivalent* (*Theorem A*) *to a system obtained from* $\sum$ *by replacing the* x*'s by* 0 *or* 1; *and let* $\Phi(a, b)$ *be an arithmetic function equivalent to the system obtained from* $\sum$ *by replacing* (1) *the* x*'s by* 0 *and* (2) *the non-*x*'s by* 1; *then the function*

$$F(a, b) \equiv f(a, b) + \frac{0}{\Phi(a, b)}$$

is an arithmetic equivalent of $\sum$.

For $F(a, b) = f(a, b) + (0/1)$ when a, b are such that $a \oplus b = 0$ or 1 in $\sum$; and $F(a, b) = f(a, b) + (0/0)$ when a, b are such that $a \oplus b = x$ in $\sum$.

By defining a/b as the (unique) element y which satisfies the boolean equation $by = a$, we can obtain boolean representations of systems $\sum$ of Theorem *C*. Since 0/0 is not an element of a boolean algebra, we have

THEOREM *C'*. (The same as Theorem *C* with the word *arithmetic* everywhere replaced by the word *boolean*.)

Thus, to find a representation of the system $\sum \equiv (K, \oplus)$ determined by the table

$$\begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & x \end{array} \qquad (x \text{ not in } K),$$

take from Table A system 7† for $f(a, b)$ and system 15 for $\Phi(a, b)$. An arithmetic representation of $\sum$ is then

$$a + b \pmod 2 + \frac{0}{ab + 1 \pmod 2},$$

* I am using the expression "$a \oplus b = x$ not in K" for some values a, b, to mean that for the values a, b in question $a \oplus b$ is meaningless.

† Or also system 8.

and a boolean representation is

$$ab' + a'b + \frac{0}{a' + b'}.$$

5. *Applications.* The representations of tables A, B obviously have a bearing on postulational questions. I point out here a few facts obtained with the help of the tables.

(1) A two-element boolean algebra (K, $\oplus$, $\odot$) may be represented by an arithmetic system; namely, by

$$K\!:0, 1;\quad a \oplus b = ab + a + b\,(\mathrm{mod}\,2);\quad a \odot b = ab\,(\mathrm{mod}\,2),$$

or, dually, by

$$K\!:0, 1;\quad a \oplus b = ab\,(\mathrm{mod}\,2);\quad a \odot b = ab + a + b\,(\mathrm{mod}\,2).$$

(2) The two-element boolean algebra (K, $<$), expressed in terms of the relation $<$ of inclusion, may be represented by the arithmetic system

$$K\!:0, 1;\quad a < b = ab + a = 0\,(\mathrm{mod}\,2).$$

(3) Each of the $\oplus$-tables 7 and 10 of Table A satisfies the conditions for an abelian group.* Hence, the logical elements 0, 1 form an abelian group with respect to each of the operations

$$ab' + a'b, \qquad ab + a'b'.\dagger$$

(4) In Table A, $\oplus$-tables 7 and 2 together define a field (K, $\oplus$, $\odot$).‡ Hence the logical elements 0, 1 form a field with respect to the pair of operations $ab' + a'b$, ab, and also with respect to the dual pair $ab + a'b'$, $a + b$.

(5) All boolean operations can be defined in terms of each of the operations $a'b'$, $a' + b'$.§ Hence all the arith-

* For postulates for an abelian group see TRANSACTIONS OF THIS SOCIETY, vol. 4 (1903), p. 27.

† It can be shown that the totality of the elements of *any* boolean algebra, finite or infinite, form an abelian group with respect to each of these operations.

‡ For postulates for fields see TRANSACTIONS OF THIS SOCIETY, vol. 4 (1903), p. 31.

§ See Sheffer's postulates for boolean algebras, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), p. 481.

metic operations of Table A can be defined in terms of each of the operations $ab + a + b + 1 \pmod 2$, $ab + 1 \pmod 2$. And any two-element algebra can be defined by means of an appropriate postulate-set involving the single operation 9 or the single operation 15.

(6) Huntington's abstract systems (K, $\oplus$, $\odot$) proving the independence of his first set of postulates for boolean algebras* may be replaced by the arithmetic systems† of the following table (Systems I_a', I_b', ..., VI' are independence systems respectively for postulates I_a, I_b, ..., VI).

TABLE C

	K	$a \oplus b$	$a \odot b$
I_a'	0,1	$a+b \pmod 2 + \dfrac{0}{ab+1 \pmod 2}$	ab
I_b'	0,1	ab	$a+b \pmod 2 + \dfrac{0}{ab+1 \pmod 2}$
II_a'	0,1	0	ab
II_b'	0,1	ab	0
III_a'	0,1	a	ab
III_b'	0,1	ab	a
IV_a'	0,1	$a + b \pmod 2$	ab
IV_b'	0,1	ab	$a + b \pmod 2$
V'	0,1	$ab + a + b \pmod 2$	$ab + a + b \pmod 2$
VI'	0	$0 + 0$	$0 \cdot 0$

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* See TRANSACTIONS OF THIS SOCIETY, vol. 5 (1904), p. 288.

† With the help of Table A we can, of course, write down boolean systems for Huntington's abstract systems.