REPORT ON CURVES TRACED ON ALGEBRAIC SURFACES*

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1. *Introduction.* An extensive literature to which we propose to devote this report has grown up around the following question and the related transcendental theory. *Given an algebraic surface, what can be said in regard to the distribution of its continuous systems of algebraic curves?* † For various reasons we have chosen in place of the chronological presentation one in which function theory and analysis situs play the predominant part, and that has been made possible by two papers of Poincaré (*s*, I, II). We must however recall at the outset that the general answer to the above question had been given earlier by Severi (*u*, V, VI), his methods being largely of an algebro-geometric nature (see § 14), except for the use of a very important transcendental theorem due to Picard (*q*, II, p. 241). In favor of the methods which dominate this report, it must be stated that they alone made possible the solution of some important problems, and furthermore have notably enriched the theory.

A question similar to the above may be asked concerning algebraic varieties, but in order to remain within proper bounds, we have deemed it best to omit them altogether.

2. *Connectivity Indices.* We shall have occasion to consider throughout a basic n-dimensional manifold W_n, ‡ usually an algebraic curve ($n = 2$), or a surface ($n = 4$). A sum of closed, k-dimensional, two-sided, analytic manifolds in W_n is called a *k-cycle* of the manifold, and shall be denoted by Γ_k. If it bounds, it is a zero-cycle; the fact being expressed by a homology: $\Gamma_k \sim 0 \bmod W_n$. Homologies can be added

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† The properties of linear and continuous systems, so successfully investigated by Castelnuovo, Enriques, and Severi, have been described by them in three readily accessible reports (*q*, II, p. 485; *e*; *u*, XI). These and similar references refer to the bibliography at the end of the report.

‡ For a more extended topological discussion see *n*, II, § 1; *v*, Ch. 4.

and also multiplied (but not divided) by integers. If several k-cycles satisfy no homology they are *independent.* The maximum number R_k of such cycles is the kth *connectivity index* (*linear index* for $k = 1$). The index R_1 of an algebraic variety is always even. For an algebraic curve, $\frac{1}{2}R_1 = p$ is the *genus,* for an algebraic surface or variety, $\frac{1}{2}R_1 = q$ is the *irregularity* (n, II, p. 233; n, VII, Ch. 2).

3. *Abelian Integrals.* We recall that by an *Abelian integral* attached to the algebraic curve C, of order m and genus p, whose equation is $f(x, y) = 0$, is meant an integral

$$\int R(x, y)dx,$$

where R is rational in the coordinates of a variable point on C. A *period* of it is its value taken over a cycle Γ_1 of C. The integral is of the *first kind* when it is holomorphic everywhere on the curve. It is then of the form

$$\int \frac{Q(x, y)dx}{f_y'},$$

where Q is an adjoint polynomial of order $m - 3$.* The maximum number of linearly independent † integrals of the first kind is equal to p (see r). Let $u_1, u_2, \cdots, u_p$ be a set of such integrals, and consider the equations

$$\sum_{h=1}^{n} \int_{A}^{A_h} du_i = v_i, \qquad (i = 1, 2, \cdots, p),$$

where the v's are constant, and the unknowns are the upper limits A_h on C.

(a) When there is more than one solution there are an infinite number, and the most general one depends linearly upon $r \geqq n - p$ parameters (Abel; Clebsch; c, p. 395). Their totality constitutes a so-called *linear series* on C.

(b) For $n = p$ the solution is in general *unique* (inversion in the sense of Jacobi; see c, p. 463). In the exceptional case, a certain number $p - p'$ of the points may be assigned arbitrarily, and the remaining p' are then uniquely determined.

* When C has no other singularities than double points with distinct tangents, an adjoint polynomial is one which vanishes at the double points. A similar definition holds for surfaces, with the double curve in place of the double points. See r.

† In the sense that no linear combination of them is constant.

4. *Algebraic Surfaces.* We shall denote our surface by F, take for its equation $f(x, y, z) = 0$ and assume it irreducible, of order m, with the genus of a generic plane section (to be denoted by H) equal to p. The special sections cut out by planes $y = Ct$ shall be called H_y. We shall furthermore assume F with only ordinary singularities * (double curve with triple points in finite number) and in arbitrary position as to the axes and the plane at infinity. The section H_y will then be of genus p unless its plane is tangent to F, when the genus is lowered to $p - 1$. The values $a_1, a_2, \cdots, a_n$ corresponding to contact are its *critical* values and n is the class of F. To each a_k is attached an important cycle Γ_1 of F: it is the cycle δ_k which tends to a point when y approaches a_k (vanishing cycle). When y turns around a_k, the increment of any Γ_1 of H_y is homologous to a multiple of δ_k mod. H_y (Picard, q, I, p. 95; n, VII, Ch. 2).

5. *Abelian Sums attached to Algebraic Curves.* A notable achievement it was of Poincaré's to have discovered a set of simple expressions that characterize continuous systems of algebraic curves on F (s, II, p. 56). By taking advantage of certain results known before his work but which follow readily also from his methods, we shall obtain easily a set of somewhat simpler expressions that will suffice for our purposes (n, IV, p. 343; u, IX, p. 204).

To H_y are attached p linearly independent integrals of the first kind of the particularly simple type

$$u_j = \int \frac{Q_j(x, y, z)}{f_z'}\, dx \qquad (j = 1, 2, \cdots, p),$$

where the Q's are adjoint polynomials of order $m - 3$ in x and z; however q of them, say the last q, are of degree $m - 2$ in x, y, and z, while the remaining $p - q$ of them are still of degree $m - 3$ when y is taken into consideration. Furthermore the periods of u_{p-q+l} with respect to the vanishing cycles δ_k are all zero.†

* Any F can be birationally reduced to that particular kind (Beppo Levi, o; Chisini, f).

† The integrals u_{p-q+l} are what q linearly independent integrals of total

Let now $A_1, A_2, \cdots, A_m$ be the fixed points of H_y (all at infinity), and let C be an arbitrary algebraic curve on F. The curve C may go through some A's with varied multiplicities. Let $B_1, B_2, \cdots, B_\nu$ be a group of points on H_y, including all variable points of intersection of C and H_y, and some of the A's, each taken with any multiplicity whatever. Let us then consider the sums

$$v_j(y) = \sum_{k=1}^{\nu} \int_{A_1}^{B_h} du_j, \qquad (j = 1, 2, \cdots, p). \tag{1}$$

From the behavior of the v's, we find the relations

$$\begin{cases} v_j(y) = \sum \dfrac{\lambda_k}{2\pi i} \displaystyle\int_A^{A_k} \dfrac{\Omega_{jk}(Y)dY}{Y - y}, & (j = 1, 2, \cdots, p - q), \\ v_{p-q+l}(y) = \alpha_l, & (l = 1, 2, \cdots, q), \end{cases} \tag{2}$$

where the α's are constants, the λ's are integers, and Ω_{jk} is the period of u_j with respect to δ_k. These expressions are basic for what follows.

6. *Existence Theorem and its Interpretation. Under what circumstances can the v's given by* (2) *yield, by means of* (1), *points B_h with an algebraic curve C for locus?* Assign to all but $p' \geqq p$ of the B's the position A_1, and solve for the remainder as in § 3, p' having the same meaning as there. On discussing the equations obtained (Poincaré, *s*, II, p. 75; *s*, III, p. 41; Lefschetz, *n*, VII, Ch. 4; Severi, *u*, IX, p. 278), the following necessary and sufficient conditions are obtained:

(*a*) The v's must be regular at $y = a$.

(*b*) Let there be constants β_h such that

$$\frac{1}{y - y_0} \sum \beta_h Q_h(x, y, z) \equiv Q(x, y, z) \tag{3}$$

is finite when y approaches y_0. Similarly, the expression

$$\frac{1}{y - y_0} \sum \beta_h v_h(y)$$

must equally remain finite when y approaches y_0.

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differentials of the first kind of F (see § 11) become when y is held constant. The possibility of subdividing the u's into two groups as here indicated is based upon the fundamental result recalled (loc. cit.), together with a theorem proved by Picard (*q*, II, p. 437) and Severi (*u*, VIII).

Conditions (a) and (b) are unsatisfactory notably in that they involve the particular variable y. From them may be derived others of a more significant nature and in particular not involving y (Lefschetz, n, III; n, IV, p. 345; n, VII, Ch. 4). The polynomial Q in (3) is adjoint of order $m-4$, hence

$$\int\int \frac{Q(x,y,z)}{f_z'}\,dxdy \tag{4}$$

is finite everywhere on F. In the usual terminology we say that it is of the first kind (Noether, p, I). As a matter of fact there is a relation (3) for every Q, hence (4) is arbitrary of its kind. We then have the following theorem.

In order that there may correspond to the (v)'*s an algebraic curve* C, *there must exist a cycle* Γ_2 *with respect to which the period of every double integral of the first kind is zero.**

The cycle Γ_2 is readily described. Let Δ_k be the locus of δ_k as y describes aa_k. Picard (q, II, p. 335) and Poincaré (s, II, p. 57) for finite cycles and Lefschetz (n, IV, Ch. 3) for any cycle, have established the homology

$$\Gamma_2 \sim \sum \lambda_k \Delta_k + \text{part of } H_a, \text{ mod } F.$$

When C exists we have (n, IV, Ch. 4)

$$C \sim \sum \lambda_k \Delta_k + \text{part of } H_a, \text{ mod } F,$$

the λ's being the same integers as in (2). This brings out clearly the relation between them and the curve.

7. *Equivalence of Curves.* A variable curve C of intersection of F with a surface whose equation contains r linear parameters gives rise to a *linear system of dimension* r, denoted by $|C|$. This system is *complete* if its curves do not belong to another of dimension $> r$, as shall be assumed throughout. The complete system determined by a given curve is unique.

If the irregularity q is greater than zero, the surface contains continuous systems whose curves do not all belong to one and the same linear system (e, p. 707). We denote by

* A noteworthy feature of this theorem is that it is the only one known concerning periods of double integrals of the first kind. This is in striking contrast with Abelian integrals of the first kind for the periods of which an extensive body of theorems has long been available.

$\{C\}$ the amplest system generated by all variable curves that may approach C. It contains all systems $|C|$ determined by its curves, and may be thought of as an algebraic variety whose elements are linear systems. Indeed when $\{C\}$ is sufficiently general, this variety is an Abelian variety of genus q, the so-called Picard variety of F.

By $|C+D|$ and $\{C+D\}$ are meant the systems defined by the composite curve $C+D$. They are uniquely determined by $|C|$ and $|D|$ in the first case, and by $\{C\}$ and $\{D\}$ in the second case. If D is in $|C|$ (or in $\{C\}$) we write $|2C|$ (or $\{2C\}$) for the sum. The meaning of $|kC|$ and of $\{kC\}$ follows immediately, if k is a positive integer. Likewise the meaning of $|C-D|$ and of $\{C-D\}$ is immediate (see *h*). The last two systems may not actually exist; they are then called *virtual.** Although not represented by any geometric configuration, there are important related symbols, and the introduction of these systems (Severi, *u*, X) has been very useful. Similar remarks hold for $|k_1C_1+k_2C_2+\cdots+k_\nu C_\nu|$ and $\{k_1C_1+k_2C_2+\cdots+k_\nu C_\nu\}$, where the k's are integers.

The v's are the same for all curves of a linear system; and for curves of a continuous system, they differ in the constants α, but not in the integers λ. In any case it is clear that the v's of the same index are combined like their curves.†

A partial result of Poincaré's (*s*, II, p. 98) which I have completed is the following theorem (*n*, IV, Ch. 4).

If C and D have the same v's, then D belongs to $|C|$. If their integers λ are the same, i.e., if only the first $p-q$ functions v are the same, then there is a positive integer k such that $\{C+kH\}$ and $\{D+kH\}$ coincide.

This justifies the following definition: C and D are *equivalent,* and we write $C=D$ if there is an E such that $\{C+E\}$ and $\{D+E\}$ coincide. A meaning is then readily attributed

* Moreover, even when $\{C-D\}$ exists it is not necessarily unique.

† Henceforth we mean by the set of v's corresponding to C that obtained when all points of intersection of C with H_y are considered and no others.

to a relation

$$\mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_\nu C_\nu = 0.$$

These notions go back essentially to Severi (u, V, p. 198). However, his definition of equivalence is somewhat narrow. The question has been searchingly examined by Albanese and our definition corresponds to what he calls *virtual equivalence* (b, p. 165).

8. *Identity with Homology.* Much interest is added to what precedes by this proposition (Lefschetz, n, IV, p. 347; n, VII, Ch. 4): *There is complete equivalence between the two relations*

$$\sum \mu_i C_i = 0; \qquad \sum \mu_i C_i \sim 0, \bmod F.$$

Thus a purely algebro-geometric notion is reduced to one of analysis situs. Well known propositions from the latter theory yield at once these results, whose geometric content we owe to Severi (n, IV; n, VI; u, V; u, VI):

(a) *There is a positive integer ρ such that any $\rho + 1$ curves satisfy a relation of equivalence, this not being the case for some sets of ρ of them.* The number ρ is the well known Picard number (q, II, p. 241).

(b) *The number ρ is the maximum number of distinct two-cycles with respect to which every double integral of the first kind has a zero period.*

(c) *Given a curve C, of sufficiently general type,* there may exist $\sigma - 1$ others, $C_2, C_3, \cdots, C_\sigma$, such that $C_i \neq C_1, \lambda C_i = \lambda C_1$, $\lambda > 1$.* The number σ is the product of Poincaré's so-called *torsion coefficients* for linear or two-cycles † (they are the same for F).

(d) *There exists an ordinary base for the curves of F, i.e., a set of curves $C_1, C_2, \cdots, C_\rho$, such that whatever C we may write*:

$$\mu C = \mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_\rho C_\rho.$$

The term *ordinary* base was introduced by Severi. By increasing the number of curves to $\rho + \sigma - 1$, we may have what Severi terms a *minimum base*, i.e., such that $\mu = 1$ for

* If virtual curves are admitted no restrictions are needed.

† Examples of surfaces with $\sigma > 1$ have been given by Severi (u, VI), Godeaux (k) and the writer (n, IV, p. 362). On such an F can be found a Γ_1 such that $\lambda\Gamma_1$ bounds ($\lambda > 1$) while Γ_1 does not, and similarly for two-cycles.

every C. Let these curves be $D_1, D_2, \cdots, D_{\rho+\sigma-1}$. Then

$$C = \nu_1 D_1 + \nu_2 D_2 + \cdots + \nu_{\rho+\sigma-1} D_{\rho+\sigma-1}.$$

All these curves may be actually chosen *effective* (non-virtual) except for $\rho = 1$, when it may be necessary to take $\sigma + 1$ curves in the minimum base, if it be desired to have them all effective. The geometric content of this answers completely the question proposed at the beginning of our report. Indeed, let $D_1, D_2, \cdots, D_\nu$ be a minimum base composed exclusively of effective curves. For every C we may write

$$C = \sum \mu_i D_i = \sum \mu_i'' D_i - \sum \mu_i' D_i,$$

where the μ''s and μ'''s are non-negative integers. Then there exists a positive integer μ such that the continuous systems

$$\{C + \mu H + \sum \mu_i' D_i\}, \qquad \{\mu H + \sum \mu_i'' D_i\}$$

coincide (Albanese, *b*, p. 204).

9. *Integrals of the Second and Third Kinds.* Abelian integrals constitute the natural analytical tool in the study of sets of finite points on an algebraic curve, for these points appear either as the set of singular points of the integral, or as the zeros of the integrand. For an entirely similar reason, the integrals that generalize Abelian integrals are of paramount interest in investigations on algebraic curves of a surface, and thus find a proper place here. We recall that an Abelian integral of the *second kind* is one behaving everywhere on its curve C like a rational function. Finally an integral is of the *third kind* if not of one of the other two kinds. Its singularities other than poles consist in a finite number of logarithmic points and to each belongs a logarithmic period corresponding to a small circuit surrounding the point. The sum of the logarithmic periods is zero. Hence there must be at least two logarithmic points, and in fact there is an integral having any two points of C for logarithmic singularities and no other. Integrals of the second kind are linearly independent if no linear combination of them reduces to a rational function. The maximum number of such integrals is twice the genus of C (see *r*).

10. *Generalization to Double Integrals.* On passing to F there are two modes of extending Abelian integrals, first to double integrals, next and much less obviously to integrals of total differentials (Picard, q, I, p. 102), which we shall consider below.

We have defined double integrals of the first kind in § 6. On the transcendental side there is no other theorem than the one mentioned there, although the number of linearly independent integrals and the related linear system cut out by adjoint polynomials of order $m-4$ (canonical system) have played a large part in investigations on surfaces (e, p. 704), this being due of course to their invariance under birational transformations. A question which I have been able to settle only in some special cases (n, IV, p. 349) is still outstanding: *Can a double integral of the first kind be without periods?* * The answer (which is probably negative) would have an important bearing upon our subject.

In the several existing treatments of double integrals of the second kind (Picard, q, II; Lefschetz, n, II, p. 242; n, VII, Note I), the type

$$(5) \qquad \int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dxdy\,, \qquad (U,V \text{ rational}),$$

introduced by Picard, plays somewhat the same part as that played by rational functions for Abelian integrals. The reason is here again invariance under birational transformations. The following mode of attack (n, VII, Note I) seems shortest and best, especially in that it exhibits a noteworthy theorem and is readily extended to integrals of any multiplicity of a higher variety: We take two-cycles in maximum number, say $\Gamma_2^1, \Gamma_2^2, \cdots, \Gamma_2^{\rho_0}$ such that no matter how ample a set of curves $D_1, D_2, \cdots, D_r$ is given, there is a cycle $\sim \Gamma_2^i$ not intersecting any of them. The curves $C_1, C_2, \cdots, C_\rho$ of a Severi base are two-cycles independent of the Γ_2^i, and every Γ_2 depends upon the $\rho_0+\rho$ thus obtained, whence

$$\rho_0 = R_2 - \rho.$$

It is then found that J is without period with respect

* It will be recalled that the corresponding theorem for Abelian integrals is proved with ease (see r).

to $\Gamma_2{}^i$, while on the contrary any integral *not* of this type will have a period with respect to some such cycle, whence it follows that the maximum number of linearly independent integrals of the second kind is ρ_0. But it is known (Alexander, *a*; Lefschetz, *n*, II, p. 239; *n*, VII, Ch. 3) that

$$R_2 = I + 4q + 2,$$

where $I = n - m - 4p$ is the well known Zeuthen-Segre invariant. From this follows Picard's formula

$$\rho_0 = I + 4q - \rho + 2.$$

The very suggestive relation between the number of linearly independent integrals of the second kind and the number of cycles of a certain type may be extended to multiple integrals of higher varieties (*n*, VI).*

11. *Integrals of Total Differentials.* By *integral of total differentials* is meant one of type

$$\int Rdx + Sdy, \qquad \frac{\partial R}{\partial y} = \frac{\partial S}{\partial x} \cdot$$

The classification and independence theorems are as for Abelian integrals with q in place of p (Picard, *q*, I, Ch. V; Castelnuovo-Enriques and Severi, *e*, p. 715; Poincaré, *s*, II, p. 91). The extension of Abel's theorem has been the object of extensive investigations by Severi (*u*, II, III).

We are particularly interested in integrals of the third kind. Let J be one, $C_1, C_2, \cdots, C_k$ its singular curves. In the vicinity of C_i the integral behaves either like a rational function or like a logarithm. In the latter case there is a logarithmic period, and C_i is a logarithmic curve. By investigating the integral which J determines on H_y, Picard has shown that when the logarithmic curves are arbitrary, J exists, provided that their number exceeds a certain integer ρ whose first appearance in the literature was precisely in this connection (*q*, II, p. 240). That it coincides with the integer denoted by ρ in § 8 follows from the following elegant theorem due to Severi.

* As a matter of fact it holds for integrals of the second kind of all types, down to Abelian integrals.

In order that $C_1, C_2, \cdots, C_k$ *be logarithmic curves of some integral of total differentials, it is necessary and sufficient that they satisfy a relation of equivalence* (Severi, *u*, V, p. 209; Lefschetz, *n*, VII, Note I). From this and Picard's result follows our assertion as to ρ.

12. *Genera of Curves. Number of Intersections.* We shall use the following notations (*u*, V; *n*, I):

$[C]$ = genus of the generic curve of $\{C\}$;

$[CD]$ = number of intersections of C and D;

$[C^2]$ = number of intersections of two curves of $\{C\}$.

An extensive symbolic calculus may be developed for these expressions (*n*, I), based on the following two formulas:

$$[(C+D)^2] = [C^2] + 2[CD] + [D^2], \tag{6}$$

$$[(C+D)] = [C] + [D] + [CD] - 1. \tag{7}$$

The proof of (6) is immediate, and that of (7) (Noether, *p*, III; Picard, *q*, II, p. 106) may be carried out very simply as follows. Subdivide a Riemann surface for C into α_2 two-cells with α_1 edges and α_0 vertices. The expression $\alpha_0 - \alpha_1 + \alpha_2$ is independent of the mode of subdivision and when C is irreducible its value is $2 - 2[C]$. Let now C vary and acquire a new double point. With a properly chosen subdivision, it is found that α_1 alone varies, and in fact decreases by 2, so that $\alpha_0 - \alpha_1 + \alpha_2$ is increased by 2. Similarly, if C acquires d double points, the expression increases by $2d$. Let then the generic curves of $\{C\}$, $\{D\}$, $\{C+D\}$ be irreducible, which is the only case of interest, and let α_i', α_i'' be the α_i of D and of the generic curve of $\{C+D\}$. We have at once

$$2[CD] + \alpha_0'' - \alpha_1'' + \alpha_2'' = (\alpha_0 - \alpha_1 + \alpha_2) + (\alpha_0' - \alpha_1' + \alpha_2'),$$

$$2[CD] + 2 - 2[(C+D)] = 2 - 2[C] + 2 - 2[D],$$

whence (7) follows.

The formulas (6) and (7) can be greatly generalized, and in particular lead to a meaning for the characters of any system whatever, even reducible or virtual. More important, however, is the following fact. Let $C_1, C_2, \cdots, C_\rho$ be a base, and let

$$\lambda C = \sum \lambda_i C_i, \qquad \mu D = \sum \mu_i C_i;$$

then (n, I, p. 207)

$$(8) \qquad [C] = \prod_{i=1} (1 + C_i)^{\frac{\lambda i}{\lambda}} - \frac{1}{\lambda} \sum_{i=1} \lambda_i,$$

where the product at the right is to be expanded according to the binomial theorem, the terms of degree one and two alone being kept and then replaced by the corresponding character. Similarly (Severi, u, V, p. 223)

$$(9) \quad [CD] = \left[\left(\sum \frac{\lambda_i}{\lambda} G_i\right)\left(\sum \frac{\mu_k}{\mu} C_k\right)\right] = \frac{1}{\lambda\mu} \sum \lambda_i \mu_k [C_i C_k].$$

It is clear that (8) and (9) give the genera and the number of intersections of curves in terms of similar data concerning the curves of the base, when their expressions in terms of the base are known. Let, for example, F be a ruled surface, and denote a generator by G. From analysis situs (n, VII, Ch. 3) it follows readily that any $\Gamma_2 \sim \alpha H + \beta G$. In particular any algebraic curve $C \sim \alpha H + \beta G$, and therefore

$$C = \alpha H + \beta G.$$

(See also Severi, u, I, p. 22.) Since $[G_2] = 0$ we have at once $\alpha = [CG]$, number of intersections of C with any G. Then from $[H^2] = m$, $[HG] = 1$, follows $\beta = \mu - m\alpha$, where $\mu = [CH]$, order of C. In terms of α and μ we have $G = \alpha H + (\mu - m\alpha)G$. On remarking that $[G^2] = 0$, $[H] = p$ (here p is the genus of the ruled surface also), (8) gives

$$[C] = 1 + \mu(\alpha - 1) + \alpha(p - 1) - \frac{m\alpha(\alpha - 1)}{2}.$$

Finally, if

$$C' = \alpha' H + (\mu - m\alpha')G,$$

we have from (9)

$$[CC'] = \alpha\mu' + \mu\alpha' - m\alpha\alpha'.$$

Both these formulas are due to Segre (t). Observe that $G = 0$ requires $\alpha = \beta = 0$, which means that H and G satisfy no equivalence; and as they constitute a minimum base, we have $\rho = 2, \sigma = 1$.

13. *Existence Theorem for* $\{C\}$. In terms of the characters $[C]$, $[C^2]$, some noteworthy results concerning the dimension of $|C|$ or $\{C\}$ may be readily expressed. In order to avoid introducing new terms we only give a somewhat special proposition taken from the beautiful theory due mostly to Castelnuovo-Enriques, and Severi (*e*, 706):

Let $s = [C^2] - [C] + 1 - q$. *If* $s \geqq 0$, $\{C\}$ *exists and consists of* ∞^q *linear systems. Should the generic C be irreducible,* $|C|$ *is of dimension* $\geqq s$.

14. *Severi's Criterion.* In his first and most important paper on the base, appears this criterion for equivalence:

If A, B *are of the same order and if* $[A^2] = [AB] = [B^2]$ *then* $\lambda A = \lambda B, \lambda \neq 0$.

The proof outlined here constitutes a simplification of Severi's. Let $[B] \geqq [A]$, and set $H_t = H + t(A - B)$. From $[HB] = [HA]$, it follows

$$[H_t H] = [H^2] = m,$$

i.e., the order of H_t is fixed. It is then found, say by (8) and (9), that

$$[H_t^2] - [H_t] + 1 - q = m - p + t[B] - t[A] + 1 - q,$$

which approaches ∞ with t. Hence all systems $\{H_t\}$, with t above a certain limit, exist. As their curves are all of same order, the number of distinct systems among them is finite, so that, for example, $\{H_{t_1}\}$ and $\{H_{t_2}\}$, $t_1 \neq t_2$, coincide. Hence, we have at once

$$(t_1 - t_2)A = (t_1 - t_2)B.$$

From this criterion Severi concludes that in order that $\sum_{i=1}^{k} \lambda_i C_i = 0$, it is necessary and sufficient that the matrix

$$\left\| \begin{matrix} [C_1 H], & [C_2 H], & \cdots, & [C_k H] \\ [C_1^2], & [C_1 C_2], & \cdots, & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ [C_1 C_k], & \cdots, & \cdots, & [C_k^2] \end{matrix} \right\| \tag{10}$$

be of rank $< k$.

Suppose for example that C_1, C_2, $\cdots$, C_k are logarithmic curves of a certain integral of total differentials J with a period α_i with respect to C_i. On any curve D not a C_i there is determined an Abelian integral with the $[DC_i]$ intersections of C_i and D for logarithmic points, the corresponding periods being α_i. It follows $\sum \alpha_i[DC_i] = 0$. As a matter of fact D may be replaced in this relation by any curve whatever, and therefore the matrix (10) is of rank $< k$. Hence the (C)'s are related by an equivalence. Conversely if they are so related J exists and is readily formed. This is essentially Severi's proof of his theorem stated in No. 11 (u, V).

15. *Determination of* ρ *and* σ. The actual determination of these elements for a given F is a difficult problem. A regular process may be given, but it has little practical value. However, various special methods have yielded the solution in all cases of interest. These cases fall mainly into two classes which we examine in turn.*

The first class consists of surfaces of sufficiently general nature contained in known algebraic varieties. The simple varieties (linear spaces or their loci, Abelian varieties) yield the most significant results. As early as 1882, Noether stated (p, II) the following important theorem: An arbitrary non-singular surface of order $m > 3$ situated in an S_3 ($S_r =$ r-space) contains only curves that are complete intersections with other surfaces of S_3. It follows that for such a surface a plane section constitutes a minimum base, and therefore $\rho = \sigma = 1$. An incomplete proof of a geometric nature, of a similar theorem for surfaces that are complete intersections of $r - 2$ varieties in an S_r was given by Fano (j, II).†

* In e (p. 730) will be found references pertaining to surfaces not discussed here.

† He makes an assumption leading to the impossibility for a double integral of the first kind to be without periods. The proof in n, IV, p. 358, is correct but for this exception noticed by Fano: the quartic surface intersection of two quadrics in S_4. The proof (loc. cit.) fails when the integer denoted there by n is negative, and a very simple discussion shows that when r exceeds three, this occurs only in the case just mentioned.

As beyond the scope of this report but noteworthy here must be mentioned extensions to algebraic varieties (Klein, m; Fano, j, I; Severi, u, VIII; Lefschetz, n, IV, p. 359).

A very general theorem from which the preceding may be derived was given by the writer by means of analysis situs (n, IV, p. 355). The following important special case will suffice as an indication. In a three-dimensional variety V_3 let $|F|$ be a linear system of surfaces, ∞^3 at least, of sufficiently general type (more or less analogous to the system of hyperplane sections). If the number of linearly independent double integrals of the second kind of a generic F exceeds that of V_3 (see n, IV), a base for the surfaces of V_3 intersects F into a similar one for its curves. V_3 and F have then equal numbers ρ, σ. (These numbers and the bases are defined for V_3 as for a surface.) Thus, to prove Noether's theorem, it is sufficient to show that F possesses double integrals of the first kind with periods not all zero, which can be done in this case (n, IV, p. 358; n, VII, Ch. 5). These will constitute integrals of the second kind not of type (5) (Picard, q, II, p. 365).

As S_3 possesses no double integrals of the second kind that are linearly independent, the theorem becomes applicable. A plane constitutes a minimum base for S_3 and its trace on F, that is a plane section, will be one for F. The surface being regular, Noether's theorem follows readily.

The second class of surfaces for which the determination of the bases has been carried out is composed of hyperelliptic surfaces and surfaces which are the image of pairs of points of two algebraic curves.

In a series of papers dating from 1893, G. Humbert made a searching investigation of hyperelliptic surfaces (l). Applying a result due to Appell, he showed in particular that the parametric equation of any curve traced on a hyperelliptic surface F is characterized by the vanishing of an intermediary function attached to the period matrix (entire function such that the addition of periods has merely the effect of multiplying it by a linear exponential) (l, first paper). From this and through an elegant analysis, Bagnera and de Franchis obtained the value of ρ and the bases even for hyperelliptic surfaces of very special type (d).

Surfaces images of point-pairs of two algebraic curves have been investigated by various authors (*e*, p. 730), the most important contribution being Severi's (*u*, I). In substance he shows that to every correspondence between the two curves corresponds a curve on F, and vice versa. Let λ be the number of singular correspondences between the two curves in the sense of Hurwitz. Severi's result gives readily $\rho = \lambda + 2$, $\sigma = 1$. However, he did not state this explicitly (loc. cit.) for his paper antedates by three years his first one on the base.

The surfaces considered in this article furnish ideal applications for the theorem (*b*) of § 8 (*n*, VII, Ch. 4). Results already known are obtained with great ease and elegance. Moreover the same method has been the basis for the extension to Abelian varieties (*n*, IV).

16. *Conclusion.* An outstanding question is the determination of ρ and the bases when only real curves are taken into consideration. So far as we know it has been solved only for rational surfaces (Comessati, *g*) and hyperelliptic surfaces (Lefschetz, *n*, V).

Algebraic varieties which we have excluded from this report are still somewhat *terra incognita*, although some important general theorems are known (*n*, IV). Perhaps we must await further information of a purely geometric nature before much progress can be expected.

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