A QUALITATIVE DEFINITION OF THE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS *

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1. *Introduction.* The object of the present note, which was suggested by a remark of Professor Birkhoff as to possible definitions of the trigonometric functions, is to define the trigonometric and hyperbolic functions by properties which shall be simple in the sense of not involving any of the ideas of the calculus, and qualitative in the sense of not involving relations as definitely explicit as functional or differential equations.†

2. *Postulates.* The characteristic properties (or postulates) which we use in our definition are stated in terms of a linear family of functions depending on two parameters, and they so restrict the family that each of its members is of the form $rF(mx + t)$ or the sum of two such expressions, where $F(x)$ is the function we wish to define. In proving this we shall incidentally give explicit rules for constructing the function $F(x)$ in terms of the family. We assume as the characteristic properties:

I. The two-fold linear family of functions $AG(x) + BH(x)$ (where $G(x)$ and $H(x)$ are any two independent members of the family) is independent of the choice of origin and direction of the x-axis; i.e., it is identical, as a family, with that given by $AG(x + c) + BH(x + c)$ or $AG(- x) + BH(- x)$.

II. Some pair of independent members of the family, as $G(x)$ and $H(x)$, are functions continuous for at least one value of x.

To these we shall add one of the following:

III (a). No member of the family never vanishes.

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† For definitions of the trigonometric functions involving such relations, see Osgood, *Lehrbuch der Funktionentheorie*, 1920, vol. 1, pp. 571–591.

III (*b*). There exists a member of the family which never vanishes, and all members of the family which never vanish are linearly dependent on this one, i.e., constant multiples of it.

III (*c*). There exist two linearly independent members of the family which never vanish.

We shall show that for I, II and III (*a*) the family is $A \sin mx + B \cos mx$, and hence each member is of the form $r \sin (mx + t)$. For I, II and III (*b*) the family is $A + Bx$, and hence each member of the family is of the form $mx + t$. Finally, for I, II and III (*c*) the family is $A \sinh mx + B \cosh mx$, and hence each member is of the form $r \sinh (mx + t)$; $r \cosh (mx + t)$ or $r \sinh (mx + t) \pm r \cosh (mx + t)$. Thus, after showing how to pick out the particular pair of functions desired from the family, we may define the trigonometric or hyperbolic functions by adjoining III (*a*) or III (*c*) respectively to I and II. It is interesting to note that as III (*a*), (*b*) and (*c*) are mutually exclusive, the conditions I and II alone define either the trigonometric functions, the hyperbolic functions, or the linear function, the last of which may be considered as a limiting case of either type. This result is natural, in view of the fact that I and the assumption that our functions admit two derivatives would practically restrict them to be solutions of a linear homogeneous differential equation of the second order with constant coefficients. It is interesting, however, to see that the much weaker restriction II is sufficient to give the result stated.

3. *Consequences of* I *and* II. Before proceeding to the separate cases mentioned above, we shall derive from I and II the fact that each member of the family is continuous at all points. To see this, let $G(x)$ and $H(x)$ be the functions mentioned in II, and let them be continuous at x_1. Then

$$G(x + x_2 - x_1) = A_1 G(x) + B_1 H(x),$$

since it is a member of the family by I, where x_2 is any number and A_1 and B_1 are constants depending on $x_2 - x_1$. But since the right member of this equation is continuous at $x = x_1$, the left member is continuous there, and $G(x)$ is continuous at x_2, any point. Similarly $H(x)$, and therefore every member of the family, is continuous everywhere.

4. *Consequences of* I, II, III (*a*). We now assume I, II and III (*a*). Let $F(x)$ be some member of the family (not identically zero) and let $F(b)$ be different from zero. We set

$$\frac{F(b+x)+F(b-x)}{2F(b)} = C(x). \tag{1}$$

It follows from this that $C(x) = C(-x)$; and that $C(0) = 1$. $C(x)$ vanishes by III (*a*), and as it is continuous, its zeros form a closed set. Let p be the smallest positive value of x for which $C(x)$ vanishes; then $C(p) = C(-p) = 0$ and we set:

$$C(x-p) = S(x). \tag{2}$$

From this we see that $S(0) = 0$; $S(p) = 1$; $S(2p) = 0$. Since $S(x)$ and $C(x)$ are not zero simultaneously, they are independent functions, and therefore every member of the family is a linear combination of them. But, by I, $S(x+h)$ and $C(x+h)$ are members of the family, and we may set

$$S(x+h) = AS(x) + BC(x), \tag{3}$$

$$C(x+h) = A'S(x) + B'C(x). \tag{4}$$

Putting in succession $x = 0$, $x = p$ in (3), and taking account of the particular values previously found, we obtain:

$$S(h) = B, \qquad S(h+p) = C(h) = A, \tag{5}$$

and (3) becomes:

$$S(x+h) = S(x)C(h) + S(h)C(x). \tag{6}$$

We may use this to obtain the value of $S(-p)$. For if we put $h = -x$, we obtain:

$$0 = S(x-x) = S(x)C(-x) + S(-x)C(x),$$
$$0 = C(x)[S(x) + S(-x)].$$

Consequently, $S(x) = -S(-x)$ for all values of x such that $C(x) \neq 0$, and hence, by the definition of p, for all values of x such that $-p < x < p$. But, since $S(x)$ is continuous at all points, we have also

$$S(-p) = -S(p) = -1.$$

This enables us to evaluate the coefficients in (4) by putting in succession $x = 0$, $x = -p$. This gives

$$C(h) = B', \qquad C(h - p) = S(h) = -A', \tag{7}$$

and (4) becomes

$$C(x + h) = C(x)C(h) - S(x)S(h). \tag{8}$$

If we put $h = -x$ in (8), we find (at least for values of x numerically less than p)

$$1 = C^2(x) + S^2(x). \tag{9}$$

Consider now a value of x between 0 and p. For such a value, we know from the continuity of the functions and the definition of p that $S(x/2)$ and $C(x/2)$ are both positive. Consequently, by combining (8) (for $h = x$) and (9), we may obtain the formulas:

$$S(x/2) = \sqrt{\frac{1 - C(x)}{2}}, \qquad C(x/2) = \sqrt{\frac{1 + C(x)}{2}}. \tag{10}$$

Combining these relations with (6) and (8), and the values obtained above, $S(0) = 0$, $S(p) = 1$, $C(0) = 1$, $C(p) = 0$, we may calculate by a finite number of operations the values of $S(x)$ and $C(x)$ for any value of x, which divided by p gives a proper fraction expressible as a terminating decimal in the binary scale. Again, since the relations and values used are all satisfied by $S(x) = \sin(\pi x/2p)$, $C(x) = \cos(\pi x/2p)$, the values so obtained will agree with the values for these functions. But, since $S(x)$ and $C(x)$ are continuous, and are equal to these functions for a set of values everywhere dense in the interval $(0, p)$, they must be identical with these functions throughout the entire interval.

Finally, by using the relations (6) and (8), and taking for h, p or $-p$ (recalling that $C(-p) = 0$, $S(-p) = -1$), we may show that if the functions given above represent $S(x)$ and $C(x)$ in any interval, they represent them in an interval longer by p in each direction, and hence for all values of x. This justifies our contention that under assumptions I, II and III (a) the family $AG(x) + BH(x)$ is of the form $A \sin mx + B \cos mx$. Furthermore, if the family is given, we may determine $S(x)$ and $C(x)$ as above, as well as p, and then define $\sin x$ and $\cos x$ by the relations: $\sin x = S(2px/\pi)$; $\cos x = C(2px/\pi)$.

5. *Consequences of* I, II, III (b). We next deduce the consequences of I, II, III (b). If $F(x)$ is the non-vanishing solution given by III (b), $F(x+c)$ and $F(-x)$, which are members of the family by I and evidently never vanish, must be linearly dependent on $F(x)$, giving:

$$F(x+c) = kF(x); \quad F(-x) = jF(x). \tag{11}$$

As the second equation reduces to $F(0) = jF(0)$ for $x = 0$, we see that $j = 1$. Combining this result with the first, we have, taking $c = (x_1 + x_2)/2$,

$$\begin{aligned} F(x_1) = F\left(\frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}\right) &= kF\left(\frac{x_1-x_2}{2}\right) \\ &= kF\left(\frac{x_2-x_1}{2}\right) \\ = F\left(\frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}\right) &= F(x_2). \end{aligned} \tag{12}$$

Since x_1 and x_2 are arbitrary, this shows that the non-vanishing solution $F(x)$ is a constant, and we may take the function $F(x)$ equal to 1.

If $E(x)$ is some non-constant member of the family, it must take on both positive and negative values, since otherwise by adding or subtracting a constant, we should obtain a second member of our family which never vanishes. Hence it must vanish at some point in such a way that the function is positive for all values of the variable in some left (or right) neighborhood of this point, and that in the right (or left) neighborhood of the same length this is not the case. Let b be the value of x at this point, and set

$$P(x) = E(b+x), \tag{13}$$

so that $P(0) = 0$; and note that all members of the family may be expressed in the form $A_1P(x) + B_1$. Then, in particular,

$$P(-x) = jP(x) + k; \tag{14}$$

and, on putting $x = 0$, we see that k is zero, and $P(-x) = jP(x)$. Since

$$P(x) = jP(-x) = j^2P(x), \tag{15}$$

we see that $j = +1$ or -1, and it cannot be $+1$, from the way in which b was selected, for this value used with (13) and (15) would give

$$E(b + h) = E(b - h),$$

and hence the behavior of $E(x)$ would be the same on the right and left neighborhoods of b. Hence $j = -1$, and

$$P(x) = -P(-x). \tag{16}$$

To obtain the addition theorem for $P(x)$, we notice from I that

$$P(x + h) = AP(x) + B. \tag{17}$$

On taking in succession $x = 0$, and $x = -h$, using (16) in the latter case, we obtain:

$$P(h) = B, \qquad 0 = P(0) = -AP(h) + B. \tag{18}$$

These relations show that $A = 1$, and reduce (17) to

$$P(x + h) = P(x) + P(h). \tag{19}$$

If we set $P(1) = m$, we see from (19) and (16) that, for rational values of x, the values of $P(x)$ equal those of mx. Hence, from the continuity of the function, we have $P(x) = mx$ for all values. Therefore, we have proved that the family is $A + Bx$ in this case.

6. *Consequences of* I, II, III (c). Finally let us consider the consequences of I, II and III (c). Let $F(x)$ be a non-vanishing member of the family, and let c be a number such that

$$F(x + c) \neq kF(-x + c). \tag{20}$$

There must be some choice of the function $F(x)$ and c, for if (20) were not true for some value of c, the reasoning used at the beginning of the discussion of the last case would show that $F(x)$ reduced to a constant, and if this held good for all the non-vanishing solutions, they would all be linearly dependent, and III (c) would be violated. We set

$$\begin{aligned} \frac{F(x + c) + F(-x + c)}{2F(c)} &= C(x), \\ \frac{F(x + c) - F(-x + c)}{K} &= S(x), \end{aligned} \tag{21}$$

where K is a constant to be determined later. It follows from the definitions that

$$(22)\qquad \begin{aligned} C(0) &= 1, & S(0) &= 0, \\ C(x) &= C(-x), & S(x) &= -S(-x). \end{aligned}$$

As before, we obtain the addition theorems by noting that

$$(23)\qquad S(x+h) = AS(x) + BC(x),$$

$$(24)\qquad C(x+h) = A'S(x) + B'C(x),$$

and evaluating the coefficients. On putting $x = 0$ in (23), we find, by using (22), that

$$(25)\qquad S(h) = B;$$

while, on putting $x = -h$, we find

$$(26)\qquad 0 = S(h-h) = -AS(h) + BC(h).$$

The last two equations show that

$$(27)\qquad 0 = S(h)[C(h) - A],$$

and hence that $A = C(h)$ when $S(h) \neq 0$. Thus, with this restriction, (23) becomes

$$(28)\qquad S(x+h) = S(x)C(h) + S(h)C(x).$$

We remove the restriction by noting that if $S(h) = 0$, while $S(x)$ is not 0, we may interchange the roles of x and h; finally, if $S(x) = S(h) = 0$, (23) and (25), which hold in all cases, show that $S(x+h) = 0$, and thus (28) holds good in this case.

If we put $x = 0$ in (24), we find $C(h) = B'$, and (24) becomes

$$(29)\qquad C(x+h) = A'S(x) + C(x)C(h).$$

If we observe that A' is a function of h, interchange x and h and subtract, we find

$$(30)\qquad A'(h)S(x) = A'(x)S(h).$$

This is an identity in x and h. If we select a value of x, x_1 for which $S(x_1) \neq 0$, and put $A'(x_1)/S(x_1) = k$, we see that $A'(h) = kS(h)$, and (29) becomes

$$(31)\qquad C(x+h) = C(x)C(h) + kS(x)S(h).$$

The number k is positive, negative, or zero. The last case cannot occur, since then we should have

$$(32)\qquad C(x+h) = C(x)C(h),\ 1 = C(h-h) = C^2(h),\ C(h) = 1.$$

But this would show that $C(h)$ was a constant, and would reduce (28) to (19), which would prove the family to be

$A + Bx$. This would not satisfy III (c). Also k cannot be negative ($= -k'$), for then we should have

$$C(x + h) = C(x)C(h) - k'S(x)S(h), \tag{33}$$
$$1 = C(h - h) = C^2(h) + k'S^2(h).$$

The second equation shows that for a value of h for which $S(h) \neq 0$, $C(h)$ is less than 1. But $C(h)$ is always positive, since it never vanishes. Since it is continuous, it takes on all values between this value and unity. Since $\cos(\pi/2^n)$ is nearer unity than any fixed number for some value of n, we may find an n and an h_1 such that

$$C(h_1) = \cos(\pi/2^n), \tag{34}$$

and since, as a consequence of (33), we have

$$C(2h) = 2C^2(h) - 1, \tag{35}$$

which is a relation satisfied by the cosine, we see that

$$C(2^{n-1}h_1) = \cos(\pi/2) = 0, \tag{36}$$

which shows that $C(h)$ is not a non-vanishing solution, and hence that the assumption that k is negative leads to a contradiction.

Since k is positive, and becomes k/m^2 if the factor K in the definition of $S(x)$ is replaced by K/m, it may be made unity by a suitable choice of K, and (31) becomes

$$C(x + h) = C(x)C(h) + S(x)S(h). \tag{37}$$

For $h = -x$, this becomes

$$1 = C(x - x) = C^2(x) - S^2(x), \tag{38}$$

while for $h = x$, we get, using (38),

$$C(2x) = 2C^2(x) - 1, \qquad C(x/2) = \sqrt{\frac{1 + C(x)}{2}}, \tag{39}$$

where the sign is determined by the fact that $C(x)$ is always positive. Let $C(x_1)$ be a value of $C(x)$ different from unity, and hence greater than unity by (38). Then $C(x_1) = \cosh mx_1$, for some positive m, since $\cosh x$ takes on all values greater than 1 for positive values of x. Hence we see from

$$C(0) = \cosh 0, \qquad C(x_1) = \cosh mx_1, \tag{40}$$

and from the fact that (39) is true for the function $\cosh mx$, that this function has the same values as $C(x)$ for all values of the form $2^n x_1$ or $x_1/2^n$. Also for these values, we see from (38) that $S(x) = j \sinh mx$, where j is plus or minus one, and must be the same for all such values of x, since (28) shows that

$$(41) \qquad S(2x) = 2S(x)C(x).$$

Finally, since (28) and (37) are the addition formulas for $\sinh mx$ and $\cosh mx$, we see that these functions agree with $S(x)$ (to within a sign) and $C(x)$ for all multiples of x_1 whose fractional parts are terminating decimals in the binary scale, and hence, since all the functions concerned are continuous, at all points. Thus under I, II and III (c), the family is necessarily $A \sinh mx + B \cosh mx$.

7. *Conclusions.* In conclusion, we notice that since I and II alone must determine one of the three types of families discussed, we may use any characteristic property of the types in place of III. Thus, we might replace III (a) by the assumption "Some member of the family vanishes twice," or "Every member of the family is bounded." This last statement may be extended so as to give an alternative form of the assumption III, in terms of bounded, instead of non-vanishing functions. That is, III (a), (b) and (c) above may be replaced by the following postulates:

III (a'). There exist two linearly independent members of the family which are bounded.

III (b'). There exists one member of the family which is bounded, and all other bounded members of the family are linearly dependent on this one.

III (c'). No member of the family is bounded.

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GROUPS IN WHICH THE NUMBER OF OPERATORS IN A SET OF CONJUGATES IS EQUAL TO THE ORDER OF THE COMMUTATOR SUBGROUP*

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1. *Introduction.* From the fact that the commutator quotient group is abelian, it results directly that there is no

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