TOTAL GEODESIC CURVATURE AND GEODESIC TORSION*

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In a paper on the gyroscope, presented to this Society April 23, 1921, Professor W. F. Osgood introduced the notion of the "bending" of a spherical curve. The "bending" is defined as the rate of turning, per unit length of arc of the curve, of the plane determined by the tangent to the curve and the normal to the surface. It is the purpose of this note to show that the bending of a curve on any surface is equal to

$$\sqrt{\frac{1}{\rho_g^2}+\frac{1}{\tau_g^2}},$$

where ρ_g and τ_g are the radii of geodesic curvature and geodesic torsion respectively. An expression, believed to be new, for the geodesic torsion of any curve of the surface is also derived. Since the rate of turning of the principal normal of a curve,

$$\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}},$$

where ρ and τ are the radii of curvature and torsion respectively, is called the total curvature of a curve,† it seems appropriate to replace the term "bending" by "total geodesic curvature."

Let Γ be any curve of a surface S and P be any point of Γ; let $\bar{\omega}$ be the angle measured from the positive direction of the principal normal to Γ at P to the positive direction of the normal to S at P, the angle being measured toward the positive binormal; let the direction cosines of the positive direction of the normal to S be X, Y, Z, and those of the positive directions of principal normal and binormal be respectively l, m, n and λ, μ, ν. Then the direction cosines of the normal to the plane of the tangent to Γ and the normal to S are

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† ENCYCLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, III, 3, 1, p. 82.

$$A = l \sin \bar{\omega} - \lambda \cos \bar{\omega},$$
$$B = m \sin \bar{\omega} - \mu \cos \bar{\omega},$$
$$C = n \sin \bar{\omega} - \nu \cos \bar{\omega},$$

and the total geodesic curvature, g, is given by

$$g^2 = \sum \left(\frac{dA}{ds} \right)^2 = \frac{\sin^2 \bar{\omega}}{\rho^2} + \left(\frac{1}{\tau} - \frac{d\bar{\omega}}{ds} \right)^2 = \frac{1}{{\rho_g}^2} + \frac{1}{{\tau_g}^2},$$

where s is the arc of Γ. The third member of the preceding equation is obtained by differentiating the values given for A, B, C and using the Frenet-Serret formulas.*

It is of interest to note the value of the total geodesic curvature for certain special curves of a surface. 1. The total geodesic curvature is numerically equal to the geodesic curvature when the geodesic torsion vanishes, that is, when Γ is tangent at P to a line of curvature of S; this occurs at all points of Γ when and only when Γ is a line of curvature; it is true in particular for all spherical curves, the case considered by Professor Osgood. 2. The total geodesic curvature is numerically equal to the geodesic torsion when the geodesic curvature vanishes, that is, when Γ osculates a geodesic line at P; this occurs at all points of Γ when and only when Γ is a geodesic line; for such a line total geodesic curvature, geodesic torsion and torsion are numerically equal. The total geodesic curvature vanishes for all points of a geodesic line of curvature, which is necessarily a plane curve. 3. If Γ is an asymptotic line, the total geodesic curvature is at every point equal to the total curvature This fact is evident from the definitions of the two curvatures, and appears also from the equations, true for any asymptotic line,

$$\rho = \pm \rho_g, \qquad \tau = \tau_g.$$

There exist, however, on every surface curves other than the asymptotic lines such that at every point the total geodesic curvature is equal to the total curvature. Along such a curve the following equation must be satisfied:

$$\left(\frac{d\bar{\omega}}{ds} \right)^2 - \frac{2}{\tau} \frac{d\bar{\omega}}{ds} = \frac{\cos^2 \bar{\omega}}{\rho^2} .$$

* See Eisenhart, *Differential Geometry*, pp. 17, 132, 138.

The asymptotic lines are given by the solution, $\cos \bar{\omega} = 0$. For a given surface with given parameters, u, v, this equation becomes a differential equation of the third order for v as a function of u. If on the other hand any curve is given, a "surface band," that is, the curve and the normal to a surface at each point of the curve, is determined by a solution $\bar{\omega}$ of this equation such that the given curve has the required property on any surface containing the band. If, in particular, the given curve is plane, the equation gives for the surface band $\cos \bar{\omega} = \operatorname{sech} \varphi$ where φ is the angle made by the tangent to the given plane curve with any fixed direction in the plane.

We now prove that the geodesic torsion of any curve Γ of S is given by

$$\frac{1}{\tau_g} = \sin \theta \frac{d\sigma}{ds},$$

where σ is the arc of the curve corresponding to Γ in the spherical representation of S and θ is the angle measured from the positive direction of Γ to the positive direction of the corresponding curve in the spherical representation, the angle being measured in the direction of rotation from the positive direction of the parametric curve (v), or v constant, on S to the positive direction of (u).

The direction cosines of the positive directions on Γ and on its spherical representation are respectively dx/ds, dy/ds, dz/ds and $dX/d\sigma$, $dY/d\sigma$, $dZ/d\sigma$. We find

$$\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{dX}{d\sigma} & \frac{dY}{d\sigma} & \frac{dZ}{d\sigma} \\ X & Y & Z \end{vmatrix} = \sin \theta.$$

Suppose the parametric system to be such that Γ is a curve (v) and that the system is orthogonal so that $F = 0$. We may then evaluate the determinant above, writing

$$\frac{dx}{ds} = \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u}, \qquad \frac{dX}{d\sigma} = \frac{ds}{d\sigma} \frac{1}{\sqrt{E}} \frac{\partial X}{\partial u},$$

using the formula given by Eisenhart * for $\partial X/\partial u$, replacing $-D'/H$ by $1/\tau_g$, and noting that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ X & Y & Z \end{vmatrix} = H.$$

We find

$$\frac{1}{\tau_g}\frac{ds}{d\sigma} = \sin\theta,$$

so proving the formula given. A somewhat simpler proof can be given for the numerical correctness of this result, that is, if the question of sign is disregarded. Certain properties of geodesic torsion appear clearly from the preceding formula. 1. If Γ is tangent to a line of curvature of S at P, the tangents to Γ at P and to the spherical representation of Γ at the corresponding point are parallel, and $\theta = 1/\tau_g = 0$. 2. If Γ is tangent to an asymptotic line of S at P, we have $\theta = \pm\pi/2$ and $1/\tau_g = \pm d\sigma/ds$. Since in the direction of an asymptotic line $d\sigma^2/ds^2 = -K$, where K is the total curvature of S, we have Enneper's theorem:† in the direction of an asymptotic line $1/\tau_g = \pm\sqrt{-K}$. 3. Since the spherical representation of a minimal surface is conformal, ‡ $1/\tau_g = \pm\sqrt{-K}\sin 2\psi$ for any curve on such a surface, where ψ is the angle of the tangent to the curve and the tangent to either line of curvature. If a minimal surface is applicable to a surface of revolution, every geodesic line of the minimal surface, which in the application corresponds to a meridian of the surface of revolution, cuts all the lines of curvature of one family at the same angle.§ For such a geodesic the torsion varies as $\sqrt{-K}$ since ψ is constant.

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* Loc. cit., pp. 116, 138.

† Eisenhart, loc. cit., pp. 140, 141.

‡ Loc. cit., p. 251.

§ This result was first given by E. Bour, *Théorie de la déformation des surfaces*, JOURNAL DE L'ÉCOLE POLYTECHNIQUE, vol. 39 (1862).