RULED SURFACES WITH DIRECTOR PLANES*

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It is well known that one family of asymptotic lines of a ruled surface is composed of the rulings or generating straight lines of the surface; that the four points in which a variable ruling is cut by any four members of the other family of asymptotic lines—hereinafter called the curved asymptotic lines—are in constant cross ratio.† In this paper we prove, using this theorem of the constant cross ratio, that the necessary and sufficient condition that a ruled surface have the property that the segments cut from a variable ruling by any three curved asymptotic lines are in constant ratio is that the surface have a director plane; that the necessary and sufficient condition that a ruled surface with a director plane have the property that any two curved asymptotic lines cut a constant length from a variable ruling is that the surface have a constant parameter of distribution. The latter type of ruled surface is further discussed: it is shown that these surfaces bear a close analogy to the right helicoid; that given any right helicoid the analogous surfaces are obtained by the choice of an arbitrary function of one variable and by two quadratures; some properties of the curved asymptotic lines are given; finally a property of the line of striction of a ruled surface with a director plane is proved—a property which is not however characteristic of these surfaces.

Suppose the rectangular coordinates of a point of any ruled surface are given by the equations

$$(1)\qquad \xi = x + ul, \qquad \eta = y + um, \qquad \zeta = z + un,$$

where x, y, z are the coordinates of a point of the directrix and are regarded as functions of a parameter v; l, m, n, the direction cosines of the ruling through the point v of the

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† Eisenhart, *Differential Geometry*, p. 249. The case of developable surfaces, where the two families of asymptotic lines coincide, is excluded in this statement and throughout the following discussion.

directrix, are also functions of v and u is length along the ruling measured from the directrix. The equation of any curve of the surface other than a ruling is given by writing in (1) u equal to a function of v. Suppose u_1, u_2, u_3, u_4 are four different functions of v giving the equations of any four curved asymptotic lines; then the cross ratio

$$\frac{u_1 - u_3}{u_2 - u_3} \div \frac{u_1 - u_4}{u_2 - u_4},$$

is constant. If u_1, u_2, u_3 give any three curved asymptotic lines, the equation of any such line is

$$\frac{u_1 - u_3}{u_2 - u_3} \div \frac{u_1 - u}{u_2 - u} = c, \tag{2}$$

where c is an arbitrary constant. It is evident that the necessary and sufficient condition that the ratio $(u_1 - u_3)/(u_2 - u_3)$ be constant for any three curved asymptotic lines is that (2) be satisfied by writing u equal to infinity, in other words that the necessary and sufficient condition that any three curved asymptotic lines cut from a variable ruling segments in a constant ratio is that the intersection of the surface with the plane at infinity be an asymptotic line. We proceed to show that this condition is equivalent to the requirement that the surface have a director plane. The differential equation of the asymptotic lines of any surface is

$$Ddu^2 + 2D'dudv + D''dv^2 = 0. \tag{3}$$

From (1) we find $D = 0$, so verifying the statement that the rulings, v constant, form one family of asymptotic lines on any ruled surface. We find further

$$(4)\qquad HD' = \begin{vmatrix} l' & m' & n' \\ l & m & n \\ x' & y' & z' \end{vmatrix},$$

$$HD'' = \begin{vmatrix} x''+ul'' & y''+um'' & z''+un'' \\ l & m & n \\ x'+ul' & y'+um' & z'+un' \end{vmatrix} = Lu^2 + Mu + N.$$

To find the condition for an asymptotic line in the plane at infinity we set $u = p/q$ and rewrite equation (3)

$$2D'(qdp - pdq) + (Lp^2 + Mpq + Nq^2)dv = 0.$$

In order that $q = 0$ satisfy this equation it is necessary and sufficient that

$$L = \begin{vmatrix} l'' & m'' & n'' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0.$$

That L vanish for all v is the condition that there exist a relation with constant coefficients, A, B, C, not all zero,

$$Al + Bm + Cn = 0.$$

It follows that the rulings are perpendicular to a fixed direction, that is, that the surface has a director plane. It is moreover evident that the asymptotic line at infinity is the intersection of the director plane with the plane at infinity and is therefore a straight line. It is easy to verify the property proved for any ruled surface with a director plane, at the same time finding by a single quadrature, in conformity to the general theory of the Riccati equation, the finite equation of all curved asymptotic lines when one such line is known in addition to the line at infinity. We may without restriction choose the yz plane as the director plane and also choose the parameter v as x, since we need not consider the plane surface x constant. We then write

$$l = 0, \qquad m = \sin \varphi, \qquad n = \cos \varphi,$$

and equations (1) and (4) may be written

$$(1') \quad \xi = x, \qquad \eta = y + u \sin \varphi, \qquad \zeta = z + u \cos \varphi;$$

$$(4') \quad HD' = \varphi', \qquad HD'' = u\varphi'' + y'' \cos \varphi - z'' \sin \varphi.$$

For the curved asymptotic lines (3) becomes

$$2\varphi' du + (u\varphi'' + y'' \cos \varphi - z'' \sin \varphi) dv = 0.$$

If u_1, u_2, u_3 are three solutions of this equation we find

$$(5) \qquad (u_1 - u_2)^2 = c_2/\varphi', \qquad (u_1 - u_3)^2 = c_3/\varphi',$$

so verifying the property of proportional segments. If u_1 is known, all curved asymptotic lines are given by $(u_1 - u)^2 = c/\varphi'$ where c is an arbitrary constant. The parameter of distribution of the surface (1′) is found to be $1/\varphi'$.*

Consider now a special case of ruled surfaces with director planes, surfaces such that any two curved asymptotic lines

* Eisenhart, loc. cit., p. 246.

cut a constant length from a variable ruling. It is evident from equations (5) that it is necessary and sufficient in order that the surface (1′) have the required property that φ' be constant, that is, that the parameter of distribution be constant. The case $\varphi' = 0$ may be excluded since for this value the surface (1′) is a cylinder. We may write $\varphi = x/a + b$ and by a proper choice of origin take $b = 0$. The equations of the required surfaces are

$$(1'') \qquad \xi = x, \qquad \eta = y + u \sin\frac{x}{a}, \qquad \zeta = z + u \cos\frac{x}{a}\cdot$$

The ruling x of (1″) is parallel to the ruling x of the right helicoid

$$\xi = x, \qquad \eta = u \sin\frac{x}{a}, \qquad \zeta = u \cos\frac{x}{a}\cdot$$

We remark that the surface (1″) is not developable for the total curvature is $-1/(a^2H^4)$. If we suppose that the directrix, $u = 0$, of (1″) is an asymptotic line we may prove certain properties of these surfaces, in particular of the curved asymptotic lines, u constant. This supposition imposes the condition on the functions y and z

$$(6) \qquad y'' \cos\frac{x}{a} - z'' \sin\frac{x}{a} = 0.$$

When a is given, that is, when the analogous right helicoid is given, equation (6) is satisfied by choosing y as any function of x and determining z by two quadratures. If z_1 is one function so determined, every value of z found from (6) has the form $z = z_1 + bx + c$. In geometrical terms we may say that, given the analogous right helicoid, the projection of an asymptotic line on the xy plane may be chosen arbitrarily; the projection of this asymptotic line on the xz plane is then determined except for the addition of the ordinates of an arbitrary straight line. When the two projections of one curved asymptotic line of the surface are known the projections of any other curved asymptotic line are obtained by adding to the ordinates of the projection of the former the ordinates of two sine curves of equal amplitude, each of the period of the analogous right helicoid $2\pi a$, differing in phase in the two projections by a quarter period. If the surface contains a finite straight line other than the rulings, such a line is a member of the family of curved asymptotic lines, and the equations of such a surface are

$$\xi = x, \qquad \eta = Ax + B + u \sin\frac{x}{a},$$

$$\zeta = Cx + D + u \cos\frac{x}{a} \cdot$$

The projections of all curved asymptotic lines of this surface are curves of the "sinoid" type obtained by adding the ordinates of a sine curve to those of a straight line, familiar to students of Smith and Gale, *New analytic geometry*.*

The discovery of a surface of this kind in M. C. Foster's thesis, *Rectilinear congruences related to special surfaces*, as one of the principal surfaces of a congruence having a plane as a limit surface, led to the discussion contained in this paper.

The line of striction of a ruled surface with a director plane (1′) is† $u\varphi' = z' \sin \varphi - y' \cos \varphi$. If the line of striction is chosen as the directrix, $u = 0$, we obtain the theorem: A necessary and sufficient condition that a curve on a ruled surface with a director plane be the line of striction is that the projections on the director plane of the ruling and the tangent to the curve coincide for every point of the curve. It may be proved, conversely, that if the projections on any plane of the tangent to the line of striction of a ruled surface and of the ruling through that point of the line of striction coincide for all points, then, if the surface is not developable, all rulings make the same angle with the plane of projection. It may be easily proved from the preceding theorem that for no other ruled surface with a director plane than the right helicoid is the line of striction an asymptotic line.

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* Pp. 111, 112.

† Eisenhart, loc. cit., p. 243.