ALGEBRAIC GUIDES TO TRANSCENDENTAL PROBLEMS.*

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1. *Introduction and Historical Sketch.* Up to the present time algebraic guides to transcendental problems have been employed extensively through only a small part of the range of subject matter to which they are so adapted as to yield important characteristic values in suggested theorems and in methods of proving them. This interaction between algebraic and transcendental analysis has attracted greater attention in the theory of integral equations than elsewhere. The relation between the theory of integral and of algebraic equations seems to have been first noticed by Volterra, who pointed out (TORINO ATTI, 1896, pp. 311–323) that a Volterra integral equation of the first kind may be regarded as in a certain sense a limiting form of a system of n linear algebraic equations in n variables as n becomes infinite. It is clear from Volterra's remarks in 1896 that the same is true of the Volterra equation of the second kind, though this fact was not then mentioned explicitly. In 1913 in his *Leçons sur les Équations Intégrales et les Équations Intégro-différentielles,* Volterra brings out in detail (pp. 30–33, 40–52) the connection between the algebraic theory and his equation of the second kind, and less fully (pp. 56 ff.) the connection between the algebraic theory and his equation of the first kind. He indicates (pp. 71 ff.) extensions of the method to systems of integral equations and to equations and systems with multiple integrals, and also (pp. 138 ff.) to the theory of permutable functions. (See also the preface and pp. 33, 102, 117 for remarks on the history of the subject and for references.) We shall set forth the character of the method by a brief indication of the nature of Volterra's treatment of the equation of the second kind.

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If in the equation

$$\varphi(x) = u(x) + \int_0^x K(x, \xi)u(\xi)d\xi$$

we replace the integral by the limit to which it is equal by definition, we have the relation

$$\varphi(x) = u(x) + \lim_{n\to\infty} \sum_{i=1}^{n} K(x, \xi_i)u(\xi_i)(\xi_i - \xi_{i-1}),$$

where the points $\xi_0 = 0, \xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n = x$ of the interval $(0, x)$ are so distributed that the greatest difference $\xi_i - \xi_{i-1}$ approaches zero with $1/n$. Approximating to the latter equation is the following:

$$\varphi(x) = u(x) + \sum_{i=1}^{n-1} K(x, \xi_i)u(\xi_i)(\xi_i - \xi_{i-1});$$

and in particular the system

$$\varphi(\xi_s) = u(\xi_s) + \sum_{i=1}^{s-1} a_{si}u(\xi_i), \quad (s = 1, 2, \cdots, n),$$

where $a_{si} = K(\xi_s, \xi_i)(\xi_i - \xi_{i-1})$. Now this algebraic system has a solution in the form

$$u(\xi_s) = \varphi(\xi_s) + \sum_{i=1}^{s-1} c_{si}\varphi(\xi_i), \quad (s = 1, 2, \cdots, n);$$

and reciprocally the former system affords the solution of the latter for the $\varphi(\xi_i)$ in terms of the $u(\xi_i)$. By analyzing these two systems Volterra obtains several fundamental relations between them. Then he proceeds heuristically to the limiting forms of these relations after the manner of forming an integral as the limit of a sum, thus obtaining the main principles upon which rests his solution of the given integral equation—these principles then being established *de novo* by methods suggested by the algebraic analysis.

The methods of Volterra may be extended to the case of the Fredholm equation; but an added difficulty arises in the new situation from the fact that the basic algebraic system has now a general determinant depending upon the kernel K rather than the much simpler determinants of the Volterra treatment. It was Fredholm's achievement to see how the method of Volterra could be extended so as to pass from the solution of a system of linear algebraic equations to the solution of the Fredholm integral equation of the second kind. Again in the case of Fredholm, the method was used merely as a heuristic guide for discovering the facts and suitable methods for their proof. With Hilbert arose for integral equations a marked extension of the method through his plan of deducing the results by limiting processes from algebraic propositions as the number of variables becomes infinite.

But the better-known applications to integral equations do not afford the earliest important uses of algebraic guides to transcendental problems. A limiting process having certain points in common with that of Volterra was employed by Cauchy in his proof of the existence of integrals of a system of differential equations, and has been preserved in the lectures of Moigno published in 1844. Considerably simplified and clarified by Lipschitz it stands today under the name of the Cauchy-Lipschitz method (with the extensions of Picard and Painlevé) as one of the principal means for establishing the existence of integrals of differential equations. (See, for instance, Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part 2, pp. 68–74.)

In some respects a closer but still a rather remote analogy with the work of Volterra is afforded by the unpublished method by which Sturm was led to many of the results of his great memoirs of 1836. On page 186 of the first volume of LIOUVILLE'S JOURNAL, Sturm tells us that it was from the solutions of the difference equation

$$L_i u_{i+1} + M_i u_i + N_i u_{i-1} = 0, \quad (i = 0, 1, 2, \cdots),$$

which is nothing more than a somewhat disguised special form of a system of linear algebraic equations, that he was led to the subject of his first memoir of 1836, the memoir in which his more characteristic results for differential equations are to be found; and that the method by which the latter were obtained from the former was one of passing by a limiting process from finite to infinitely small differences. He adds that he had also found for the difference equation properties which are not susceptible of being carried over to differential equations. From these remarks, which he seems never to have elaborated anywhere in his published writings, it is clear that Sturm was led to many of his classic results by a fundamental algebraic guide to the transcendental problems treated.

If one has in mind the suggestive remark of Sturm and the limiting process of Cauchy's existence proof, there is no longer any difficulty in carrying through the limiting process from the difference to the differential equation and of obtaining properties of the solution of the latter directly from those of the former. This was done in an interesting case by Porter in 1902 (ANNALS OF MATHEMATICS (2), vol. 3 (1902), pp. 55–70) more than two years before Hilbert in 1904 took a similar step for integral equations. If we remember that the difference equations used in this analysis are but condensed forms of certain systems of linear algebraic equations, we shall see that the processes which we have usually associated principally with the theory of integral equations were used heuristically in the theory of differential equations long before they were similarly employed for integral equations, and that they were used some years earlier in the former than in the latter in the way of a rigorous passage to the limit.

In Bôcher's Fifth International Congress address and in his *Les Méthodes de Sturm* (see especially chapter 2, pp. 14–42) there is brought out more clearly and more fully than by any of his predecessors the intimate relation that exists between the theory of systems of linear algebraic equations and differential equations, particularly with boundary conditions. Several important theorems for differential systems are obtained as immediate analogues of well known fundamental theorems for algebraic systems, and several intimately related aspects of the two problems are brought out forcibly by the statement of theorems common to the two and identical in their principal characteristics.

It will be observed that all the problems treated in these cases (with the exception of that of Cauchy) are linear in character. This is not accidental; it arises from two facts. Many of the profound phenomena of nature are subject to laws whose expression in mathematical form gives rise to fundamental linear problems of several kinds. Logically the simplest and historically the first to be treated in detail of the linear problems of pure mathematics are those having to do with systems of linear algebraic equations. And these hold the place of greatest importance, both on account of their simplicity and relatively complete development and on account of their suggestiveness in leading the way to transcendental linear problems which emerge from a direct consideration of the natural limiting cases of algebraic systems under the guidance of current problems of transcendental analysis.

My principal purpose in this address is to discuss two types of algebraic theorems and the transcendental problems to whose solution they lead the way. In the one class we shall have oscillation and comparison theorems, and in the other theorems of expansion in orthogonal functions and their generalizations. To one acquainted with the relevant theory of differential and integral equations, it is clear that there is already at hand a large body of doctrine having to do with certain transcendental problems in each of these domains, namely, the classic oscillation and comparison theorems of Sturm for differential equations of the second order, the Sturm-Liouville expansions, expansions by means of the biorthogonal functions arising from Birkhoff's theory of differential systems and their adjoints, and the expansion theorems arising in the theory of integral equations.

The principal algebraic theorems to which I shall direct your attention were conceived in the first place by considering what properties of certain approximating algebraic systems correspond to the properties already established for the transcendental problems which have been treated. It was not difficult to arrive at the corresponding theorems for the special algebraic systems which were involved in certain of these cases; and the theorems once in hand for the special systems were readily extended to fairly general classes of algebraic systems. With this much accomplished, one is in possession of algebraic facts suitable to serve as a guide to a large class of transcendental problems having certain analogies with the problems which suggested the algebraic theorems in the first place. The interaction thus set up between the algebraic and the transcendental problems has a useful power of leading forward to the discovery of important results.

In order to have clearly in mind the nature of the connection between a differential equation and a system of algebraic equations, we shall set up the relation in one of the simplest important cases. A differential equation, with or without boundary conditions, may be realized in an infinite number of ways as the limiting form of an algebraic system, so that there is always room for choice in setting up the system, and in fact need for care that it shall be done in a convenient way. In connection with the equation of second order

$$u''(x) + \varphi(x)u(x) = 0,$$

it is often convenient to employ the approximating equation

$$\frac{u(x + 2\delta) - 2u(x + \delta) + u(x)}{\delta^2} + \varphi(x + \delta)u(x + \delta) = 0,$$

which reduces to the differential equation when δ approaches zero, provided that u and φ are subject to appropriate conditions. This equation reduces to

$$u(x) + \{\delta^2\varphi(x + \delta) - 2\}u(x + \delta) + u(x + 2\delta) = 0.$$

If we are concerned with the original differential equation when x ranges over an interval (ab), we may take $\delta = (b - a)/n$ where n is an integer. Then, giving to x in the last equation the values $a, a + \delta, a + 2\delta, \cdots, a + (n - 2)\delta$, we have an algebraic system of $n - 1$ equations of the form

$$u(a + i\delta) + a_i u(a + \overline{i + 1}\delta) + u(a + \overline{i + 2}\delta) = 0,$$
$$(i = 0, 1, \cdots, n - 2),$$

from which to find the $n + 1$ unknown quantities $u(a)$, $u(a + \delta)$, $\cdots$, $u(a + n\delta)$. From properties of the solutions of this algebraic system one can pass back heuristically to properties of the solutions of the differential equation and then can establish these properties by a *de novo* argument suggested by the methods for dealing with the algebraic system. This special case may suggest certain principal characteristics of the general method as applied to differential equations.

The two instances given, the one just developed and that of Volterra, do not by themselves afford an adequate suggestion of the range of applicability of the method. From these cases one can see in part how it works for integral equations and ordinary differential equations. In a similar way it may be brought to bear upon the theory of difference and q-difference equations, both ordinary and partial, and the theory of partial differential equations. Moreover, if we pass from any of these cases by another limiting process of Volterra to such limiting forms as his linear integro-differential equations, or to linear integro-difference and integro-q-difference equations, or to various linear systems combining the properties of these mentioned types of transcendental equations, we shall be able to look upon any one of these directly as a limiting case of an algebraic system under some appropriate method of passing to the limit. In fact, it is probable that certain essential elements of these algebraic guides to transcendental problems can be realized in the case of any transcendental linear problem to which one may be led naturally.

2. *Algebraic Oscillation and Comparison Theorems.* Let us consider a graphical representation of the set of real constants $u_1, u_2, \cdots, u_n$ obtained in the following manner (cf. M. B. Porter, ANNALS OF MATHEMATICS (2), vol. 3 (1901), p. 56). On any convenient horizontal straight line segment, say the points s such that $a \leqq s \leqq b$, let us erect n perpendiculars, two of which are at the ends of the segment, while the other $n-2$ are evenly or unevenly distributed on the interior of the segment. Let these be marked from left to right by the numbers $1, 2, \cdots, n$; and consider them as analogous to the n coordinate axes of a space of n dimensions. Let the greatest distance between two consecutive axes be called the norm of the system of axes. On the ith axis let us take a point at a distance $|u_i|$ from the original segment, and above it or below it according as u_i is positive or negative. Having done this for each value i of the set $1, 2, \cdots, n$, join by straight line segments the point on each axis but the last to the point on the adjacent axis to the right. We thus obtain a broken line

which we shall call the graphic representation of the point $(u_1, u_2, \cdots, u_n)$ in space of n dimensions, or of the set of constants $u_1, u_2, \cdots, u_n$.

This broken line may be taken also as the graph of a continuous function $u(s)$ of the real variable s on the interval $a \leqq s \leqq b$. We shall say that this function $u(s)$ has been obtained from the set of constants $u_1, u_2, \cdots, u_n$ by linear interpolation with respect to the given n axes. The points at which (values of s for which) this broken line cuts the original line segment (viewed as the axis of s) we shall call indifferently the zeros of the set of constants or of the function $u(s)$.

Let us now suppose that we have a given real-valued single-valued function $v(s)$ of the real variable s, continuous and having a finite number of maxima and minima on $a \leqq s \leqq b$, and a set of n axes formed in the manner already indicated. The graph of this function will cut the n axes in points by means of which we may define as above a linearly interpolated function $\bar{v}(s)$. If the given function $v(s)$ is held the same, and the system of axes is subjected to successive changes, so that the norm of the system decreases and approaches zero, it is clear that the resulting sequence of linearly interpolated functions $\bar{v}(s)$ approaches as a limit the function $v(s)$. The situation thus briefly described is typical of the character of the limiting process by which we shall repeatedly pass from an algebraic system to the corresponding transcendental equation or system.

The most interesting known oscillation and comparison theorems are those which arise in connection with linear homogeneous differential equations of the second order. Certain of the most fundamental properties of such an equation, namely, linearity, homogeneity, and that property in virtue of which the general solution may be expressed linearly in terms of two linearly independent particular solutions, are also fundamental properties of the algebraic system of n equations

$$(1) \qquad \sum_{j=1}^{n+2} a_{ij}x_j = 0, \qquad (i = 1, 2, \cdots, n),$$

in the $n + 2$ unknown quantities $x_1, x_2, \cdots, x_{n+2}$, the matrix of the coefficients of this system being of rank n. Such a system possesses two linearly independent solutions u_i, v_i; and in terms of these the general solution may be written in the form $x_i = cu_i + dv_i$, where c and d are arbitrary constants. From the known theory of the differential equation, and by means of its relation to a particular form of system (1) as exhibited in the first section, we are led to certain properties of the solutions of this general system. We shall now state a few of these properties.

If we denote by Δ_{kl} the determinant of the matrix obtained from the matrix of coefficients in (1) by striking out the kth and lth columns, it may be shown without difficulty that Δ_{kl} and the determinant $u_k v_l - u_l v_k$ are both zero or neither zero, provided that when $\Delta_{kl} = 0$ we do not have the exceptional case in which $\Delta_{ml} = 0 = \Delta_{km}$ for every m of the set $1, 2, \cdots, n + 2$ except k and l. A fairly straightforward argument, based on this elementary result, leads to a proof of the following fundamental theorem.

THEOREM. *Let $\Delta_{i,\, i+1}$ for a given range R of consecutive values of the integer i be of one sign and let I denote the interval of the s-axis corresponding to this range of i in the sense of the treatment in the first paragraph of this section. Let u_i and v_i be two linearly independent solutions of the system* (1) *the matrix of whose coefficients is of rank n; and let these solutions be extended, by the method of linear interpolation employed above, to the functions $u(s)$ and $v(s)$. Then on the interval I the zeros of $u(s)$ and $v(s)$ separate each other.*

If one examines the proof (see AMERICAN JOURNAL, vol. 43 (1921), p. 84) by which the foregoing theorem is established, he will see that it depends intimately upon the fact that the determinant $u_{i+1}v_i - u_i v_{i+1}$ is of one sign on R. If one undertakes to formulate a corresponding theorem for the more general system

$$(2) \qquad \sum_{j=1}^{n+h} a_{ij}x_j = 0, \qquad (i = 1, 2, \cdots, n),$$

in the $n + h$ unknown quantities $x_1, x_2, \cdots, x_{n+h}$, where $h \geqq 2$, the matrix of coefficients being again of rank n, he will find the situation in some respects the same as before, but in other (and perhaps more important) respects he will find it far different. If we let D_i denote the determinant of the nth order matrix obtained from the matrix of coefficients in (2) by striking out h consecutive columns beginning with the ith, and if we denote by w_i the determinant of order h,

$$w_i = \begin{vmatrix} x_i^{(1)} & x_{i+1}^{(1)} & \cdots & x_{i+h-1}^{(1)} \\ x_i^{(2)} & x_{i+1}^{(2)} & \cdots & x_{i+h-1}^{(2)} \\ \cdot & \cdot & \cdots & \cdot \\ x_i^{(h)} & x_{i+1}^{(h)} & \cdots & x_{i+h-1}^{(h)} \end{vmatrix},$$

formed by means of a fundamental system of solutions $x_i^{(1)}$, $x_i^{(2)}, \cdots, x_i^{(h)}$ of (2), we shall find that for a range R of values of i on which D_i is of one sign it is true that w_i is of one sign; so that in this respect the situation is the same for all values of h.

But if we undertake to proceed further in the direction of a generalization of the theorem stated above, we find, not indeed that our steps are arrested, but that the theorem begins to lose its elegance and simplicity as soon as h is greater than 2, and that the complexity increases rapidly with increase of h. The marked simplicity for the case $h = 2$ is due in large measure to the fact that the expanded determinant w_i has but two terms when $h = 2$. It is possible to put in a variety of forms the complete generalization which emerges, but there seems to be no way in which we can proceed directly to the goal without a surrender of the elegance and simplicity of the theorem. The results which emerge are, however, not entirely without interest; in particular, they appear to point in the direction of theorems for differential equations (for instance) of general order h; but these will necessarily be rather complicated. We shall make no attempt to state any of the results here for general h, either for the case of the algebraic system or that of any of its limiting forms.

If the way of progress by such direct generalization is barred, we shall naturally seek the goal by some other means. Since the difficulty arises primarily from the fact that we have too many linearly independent solutions of (2), that is, too many arbitrary constants in the general solution of (2), let us restrict attention to a particular class of solutions in which the number of arbitrary elements is 2. For the case of the algebraic system (2) the simplest way to do this is to adjoin $h - 2$ additional independent equations of the same type; and this brings us back to the case of system (1). But in the case of a differential or difference equation, for instance, such a means is not directly open to us. However, it is true that the boundary conditions in the transcendental cases are represented in the algebraic system by linear equations (see Bôcher's *Les Méthodes de Sturm,* loc. cit.). The suggestion then is to employ, for a difference or differential equation of order h, boundary conditions (usually $h - 2$ in number) so that there shall be but two arbitrary elements in the general solution subject to these boundary conditions. In § 5, we shall exhibit certain results suggested by these considerations.

In what follows in this section, we shall suppose that the notation in (1) is so chosen that the determinants of the square matrices of orders 1, 2, 3, $\cdots$, n in the lower right-hand corner of the matrix $||a_{ij}||$ are all different from zero. Without loss of generality, they may be taken to be positive, since, if they were not so, this could be brought about by changing the sign of every coefficient in certain of the equations in (1); and therefore we take them to be positive. We *assume* further that the determinants of the square matrices of orders 1, 2, 3, $\cdots$, n in the upper left-hand corner of the matrix $||a_{ij}||$ are all different from zero. Then it is possible to reduce system (1) to a new system

$$x_{i+2} + \alpha_i x_{i+1} + \beta_i x_i = 0, \quad (i = 1, 2, \cdots, n),$$

where α_i and β_i are determinate functions of the original coefficients a_{ij}, and β_i is positive for all values $i = 1, 2, \cdots, n$. On writing $x_i = y_i u_i$, where y_1 and y_2 are positive quantities and

$$y_{2i+1} = \beta_{2i-1}\beta_{2i-3} \cdots \beta_3\beta_1 y_1, \qquad y_{2i} = \beta_{2i-2}\beta_{2i-4} \cdots \beta_4\beta_2 y_2,$$

we obtain for the u_i the relations

$$(3) \qquad u_{i+2} + \varphi_i u_{i+1} + u_i = 0, \qquad \varphi_i = \alpha_i \frac{y_{i+1}}{y_{i+2}},$$

where φ_i is a determinate function of the a_{ij} in (1). A second system of the form (1), under such hypotheses as we have just employed, would reduce to the normal form

$$(4) \qquad v_{i+2} + \psi_i v_{i+1} + v_i = 0, \quad (i = 1, 2, \cdots, n).$$

Comparison theorems for the distribution of the zeros of the functions $u(s)$ and $v(s)$, obtained from the constants u_i and v_i by linear interpolation, yield corresponding theorems for the two original systems of form (1). We state a few of the results for the normal forms (3) and (4).

THEOREM. *Let u_i and v_i be solutions of equations* (3) *and* (4), *respectively, and let $u(s)$ and $v(s)$ denote the functions into which they interpolate linearly with respect to a given system of coordinates. If $u(s)$ has consecutive zeros on the μth and $(m+1)$th intervals, $\mu < m$, then $v(s)$ has a zero between these zeros of $u(s)$ provided that either*

(*a*) $\varphi_i \leqq \psi_i$, $(i = \mu, \mu + 1, \cdots, m)$, *the equality sign not holding for all these values*; *or*

(*b*) $\varphi_i = \psi_i$, $(i = \mu, \mu + 1, \cdots, m)$, *and the sets of constants u_i and v_i, for $i = \mu, \mu + 1, \cdots, m$, are linearly independent.**

From this several further properties of comparison are readily derived. We state two of them.

Suppose that $u_1 \neq 0$, $v_1 \neq 0$, $\varphi_i \leqq \psi_i$, $(i = 1, 2, \cdots, \nu)$, and that $u_2/u_1 > v_2/v_1$. Then, if $u(s)$ has k zeros on the first ν intervals of the coordinate system, $v(s)$ has at least k zeros on these intervals and the jth of these zeros (in increasing order) of $v(s)$ is to the left of the jth one of $u(s)$.

Let u_1, v_1, u_{k+1}, v_{k+1} be all different from zero and let $u_2/u_1 > v_2/v_1$. Let $u(s)$ and $v(s)$ have the same number (which may be zero) of roots on the first k intervals. Then we have

$$\frac{u_{k+2}}{u_{k+1}} > \frac{v_{k+2}}{v_{k+1}},$$

provided that $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, k$. Other similar theorems may be stated by modifying in certain respects the inequalities in the hypothesis and the conclusion.

* Compare the related theorem due to M. B. Porter, ANNALS OF MATHEMATICS, (2), vol. 3 (1902), p. 65.

3. *Algebraic Expansion Problems.* The r homogeneous linear algebraic systems in the unknown quantities x,

$$(5)\quad \sum_{j=1}^{n_h} (a_{0hij} + \lambda_1 a_{1hij} + \lambda_2 a_{2hij} + \cdots + \lambda_r a_{rhij}) x_{hj} = 0, \qquad (i = 1, 2, \cdots, n_h),$$

a separate system being formed for each value h of the set $1, 2, \cdots, r$, and the r adjoint systems

$$(6)\quad \sum_{j=1}^{n_h} (a_{0hji} + \lambda_1 a_{1hji} + \cdots + \lambda_r a_{rhji}) y_{hj} = 0, \qquad (i = 1, 2, \cdots, n_h),$$

are consistent (in the sense that each system in each set of r systems has a solution not identically zero) for precisely the same values of the parameters $\lambda_1, \lambda_2, \cdots, \lambda_r$, namely, those values which satisfy the characteristic system of r determinantal equations

$$(7)\quad |a_{0hij} + \lambda_1 a_{1hij} + \cdots + \lambda_r a_{rhij}| = 0, \qquad (h = 1, 2, \cdots, r),$$

where for a given value of h the first member is the determinant of order n_h whose element in ith row and jth column is that which is written out explicitly.

The sets of characteristic values of $\lambda_1, \lambda_2, \cdots, \lambda_r$ for (5) and (6), namely, the sets of solutions of (7), necessarily finite in number if certain exceptional cases are avoided (as we intend they shall be), we shall denote by

$$\lambda_1^{(\rho)}, \lambda_2^{(\rho)}, \cdots, \lambda_r^{(\rho)}$$

for varying values of ρ, the two ordered sets being distinct for two distinct values of ρ. The corresponding solutions of (5) and (6) we shall then denote by

$$x_{hj}^{(\rho)}, y_{hj}^{(\rho)}, \qquad (j = 1, 2, \cdots, n_h;\ h = 1, 2, \cdots, r).$$

If we avoid certain exceptional cases, readily described in terms of the coefficients a in (5) and (6), we then have (for varying ρ and σ) the following relations expressing the fundamental conjugate character of the solutions of (5) and (6), namely:

$$(8)\quad \sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1 j_1 \cdots i_r j_r} \prod_{h=1}^{r} x_{i_h h}^{(\rho)} y_{h j_h}^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases}$$

where

$$D_{i_1j_1\cdots i_rj_r} = \begin{vmatrix} a_{11j_1i_1} & a_{21j_1i_1} & \cdots & a_{r1j_1i_1} \\ a_{12j_2i_2} & a_{22j_2i_2} & \cdots & a_{r2j_2i_2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1rj_ri_r} & a_{2rj_ri_r} & \cdots & a_{rrj_ri_r} \end{vmatrix}.$$

In what follows we shall assume that the mentioned exceptional cases are avoided, so that we shall have situations in which relations (8) are valid.

In order to bring out clearly the nature of these conditions of conjugacy, let us consider the case when $r = 1$. Systems (5) and (6) then take the form

$$(9) \quad \sum_{j=1}^{n} (c_{ij} + \lambda b_{ij})x_j = 0, \qquad \sum_{j=1}^{n} (c_{ji} + \lambda b_{ji})y_j = 0, \qquad (i = 1, 2, \cdots, n).$$

The conditions of conjugacy reduce to

$$(10) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}x_i^{(\rho)}y_j^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma. \end{cases}$$

In the more special case when $b_{ii} = b_i$, $b_{ij} = 0$ if $j \neq i$, these become

$$(11) \qquad \sum_{i=1}^{n} b_i x_i^{(\rho)}y_i^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma. \end{cases}$$

When $b_i = 1$ for every i these become merely the usual conditions of biorthogonality; and these in turn reduce to the usual conditions of orthogonality in case we have $c_{ij} = c_{ji}$ and the same solutions x and y are taken for the two systems in (9). Thus we see that (8) affords an extensive generalization of a classic elementary relation of wide usefulness.

If m is the number of sets of characteristic values for (5) and (6), and if the set of constants $z_{i_1i_2\cdots i_r}$, for i_h varying from 1 to n_h for each h, may be "expanded" in the form

$$(12) \qquad z_{i_1i_2\cdots i_r} = \sum_{k=1}^{m} c_k x_{1i_1}^{(k)} x_{2i_2}^{(k)} \cdots x_{ri_r}^{(k)},$$

where the c_k are independent of the subscripts i_h, the foregoing properties of conjugacy are available for an immediate determination of the c_k in the form

$$(13) \quad c_k = \frac{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1j_1\cdots i_rj_r} z_{i_1i_2\cdots i_r}\prod_{h=1}^{r} y_{hj_h}{}^{(k)}}{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1j_1\cdots i_rj_r}\prod_{h=1}^{r} x_{hi_h}{}^{(k)} y_{hj_h}{}^{(k)}}.$$

It is easy to obtain, in terms of the coefficients a of (5) and (6), broad sufficient conditions for the validity of the "expansion" in (12). The essential simplicity of these formulas is more readily apparent from the special case involved in equations (9) to (11). We assume that for this case m has its usual value n. Then if

$$(14) \qquad z_i = \sum_{k=1}^{n} c_k x_i{}^{(k)}, \qquad (i = 1, 2, \cdots, n),$$

we have for c_k one or the other of the values

$$(15) \qquad \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} z_i y_j{}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} x_i{}^{(k)} y_j{}^{(k)}}, \qquad \frac{\sum_{i=1}^{n} b_i z_i y_i{}^{(k)}}{\sum_{i=1}^{n} b_i x_i{}^{(k)} y_i{}^{(k)}},$$

according as we have the case of (10) or (11). If $b_i = 1$ the latter reaches the maximum of elegance, at least if equations (9) are then self-adjoint and the same solutions x and y of the two systems are taken.

In the foregoing expansion formulas the number of subscripts on the "function" to be expanded in terms of the "functions," x is equal to the number of parameters λ involved in the original system of algebraic equations. When we proceed to the transcendental limiting cases we shall see that these algebraic results are in the form best suited to applications to equations involving functions of one variable. In order to obtain a form suitable as the heuristic guide in problems involving functions of more than one variable in the original equations, it is desirable to look upon our expansion formulas in a way slightly different from that in evidence in the foregoing work. We can best bring out what is needful by considering expansion (14). Let us suppose that n is the product $\mu\nu$ of two integers. Let us replace the subscript i, running over the set $1, 2, \cdots, n$, by the double subscript ij where i

runs over the set 1, 2, $\cdots$, μ and j over the set 1, 2, $\cdots$, ν. Then equations (14) become

$$(16)\quad z_{ij} = \sum_{k=1}^{\mu\nu} c_k x_{ij}^{(k)}, \qquad (i = 1, 2, \cdots, \mu;\ j = 1, 2, \cdots, \nu).$$

The coefficients c_k are of course determined by the same methods as before but the formulas are modified through the replacing of single summations by double summations. It is clear that we may pass in a similar way to multipartite instead of bipartite subscripts. This obvious remark concerning a change from one subscript to more than one subscript entails important consequences in a great variety of expansion problems.

With a given set of n axes on an interval $a \leqq s \leqq b$, as in the opening paragraphs of § 2, let us extend by linear interpolation the n sets $x_i^{(k)}$, $k = 1, 2, \cdots, n$, and the set z_i of equation (14); and let $x^{(k)}(s)$, $k = 1, 2, \cdots, n$, and $z(s)$ denote the functions of the continuous variable s so obtained. Then it is easy to prove that

$$(17)\qquad z(s) = \sum_{k=1}^{n} c_k x^{(k)}(s),$$

where the coefficients c_k have the same values as in (14). Thus we pass from "expansions" of sets of constants to interesting expansions of a particular class of functions of a continuous variable.

Let us consider the like matter for functions of two subscripts and expansions of the form (16). For the representation of u_{ij} for $i = 1, 2, \cdots, \mu$ and $j = 1, 2, \cdots, \nu$, we shall start from ν parallel planes, one for each value of j. In each of these we use for s the same range $a \leqq s \leqq b$ and place the planes in order 1, 2, $\cdots$, ν for j, each directly in front of the preceding one and arrange the vertical axes, so that the vertical axes in any one plane are (point for point) the orthogonal projections on that plane of the axes in any other plane. Next, for fixed j we extend the u_{ij} to $u_j(s)$ by linear interpolation as before. Then for each value of s we connect the points $u_j(s)$, $j = 1, 2, \cdots, \nu$, by straight-line segments joining the consecutive points. We thus get a sort of broken surface affording the graphical representation of a function $u(s, t)$, gotten (we may say) by linear interpolation from the points u_{ij}. We call this the graphical representation of the set of constants u_{ij} with respect to the given system of axes. Let $c \leqq t \leqq d$ denote the range of t in this representation, this range evidently depending on the positions of the ν planes employed in setting up the graphical representation.

Let us replace the subscripts i, j in the functions in (16) by the continuous variables s, t in accordance with the method just indicated. Then it is easy to prove that we have

$$(18) \qquad z(s, t) = \sum_{k=1}^{\mu\nu} c_k x^{(k)}(s, t), \qquad (a \leqq s \leqq b,\ c \leqq t \leqq d),$$

where the coefficients c_k have the same values as in (16). Similar (but more complicated) extensions may be associated with (12) so that we come through to a relation of the form

$$(19) \quad z(s_1, s_2, \cdots, s_r) = \sum_{k=1}^{m} c_k x_1^{(k)}(s_1) x_2^{(k)}(s_2) \cdots x_r^{(k)}(s_r),$$

where the coefficients c_k have the same values as in (12) and the functions involved depend on continuous variables. Further generalizations may also be made in (12) and in the last formula by replacing one or more of the subscripts i_h in (12) by multipartite subscripts and by linear interpolation in the resulting relations.

In the case of the principal transcendental expansion problems the foregoing expansions in a finite number of terms are replaced by .expansions in an infinite number of terms so that difficult questions of convergence arise. In one of the most important of these transcendental problems, namely, that treated by Birkhoff (TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), pp. 373–395), the convergence questions are dealt with by the aid of a contour integral by means of which the sum of any finite number of terms of the series is readily expressed. Similar contour integrals exist for representing the sum of a finite number of terms of the series in (12) or (14). Let us consider the general case. For any given value of h let $\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the determinant in the first member of (7) and let $\Delta_{hij}(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the cofactor of the element in the

ith row and jth column of this determinant. We assume for the present purpose that systems (5) and (6) are so restricted that the solutions of each for characteristic values of the λ's is unique (except for a constant factor) and that for each h

$$\lim_{\lambda_1=\lambda_1^{(k)},\cdots,\lambda_r=\lambda_r^{(k)}} \frac{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\lambda_h - \lambda_h^{(k)}}$$

exists and is finite, where $\lambda_1^{(k)}, \cdots, \lambda_r^{(k)}$ is any set of characteristic values for (5) and (6). Then, if we avoid a certain exceptional condition not arising in the (most important) case $r = 1$, we have for the kth term of the expansion of $z_{t_1 t_2 \cdots t_r}$ afforded by (12) the value

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^r \int_{\Gamma_{1k}}\int_{\Gamma_{2k}} \cdots \int_{\Gamma_{rk}} \left\{ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1 j_1 \cdots i_r j_r} z_{i_1 i_2 \cdots i_r} G_{j_1 t_1 \cdots j_r t_r} \right\} d\lambda_1 d\lambda_2 \cdots d\lambda_r,$$

where Γ_{hk} $(h = 1, 2, \cdots, r)$ is a contour in the λ_h-plane about the point $\lambda_h^{(k)}$, containing in its interior no other characteristic value of λ_h, and where

$$G_{j_1 t_1 \cdots j_r t_r} = \prod_{h=1}^{r} \frac{\Delta_{h j_h t_h}(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)} .$$

If we replace the contours Γ_{hk} $(h = 1, 2, \cdots, r)$ by Γ_h, a contour which includes within it all the characteristic values of λ_h, and perform the same multiple integration about such contours, we shall have the value of the function $z_{t_1 t_2 \cdots t_r}$. It is clear that we may form similarly the contour integral for any given partial sum of the series for $z_{t_1 t_2 \cdots t_r}$ in (12). For the case of systems (9) we have the equations

$$z_t = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\Delta_{jt}(\lambda)}{\Delta(\lambda)} b_{ji} z_i d\lambda, \qquad (t = 1, 2, \cdots, n),$$

where Γ is a contour in the λ-plane inclosing all the characteristic values for (9), and where $\Delta(\lambda)$ is the determinant of the coefficients in the first member of (9) and $\Delta_{jt}(\lambda)$ is the cofactor of the element in the jth row and tth column of $\Delta(\lambda)$.

For use in connection with expansion problems involving difference equations, it is desirable to observe that several of the important formal properties of (5) and (6) are preserved in the case of infinite systems in which appropriate conditions of convergence are realized. Thus in (5) and (6) we may allow some or all of the numbers $n_1, \cdots, n_r$ to become infinite; for the sake of simplicity we suppose that all of them become infinite. The system (7) is replaced by a transcendental system having, in the cases which interest us, an infinite number of solutions. Subject to suitable limitations for procuring convergence, we may proceed as before and thus derive a new form of (8) in which the change is that of replacing each of the numbers $n_1, \cdots, n_r$ by ∞, and a new form of (12) in which m is replaced by ∞ and the relations are to be valid for each i_h ranging over the set $1, 2, 3, \cdots$, the expressions for the coefficients c_k $(k = 1, 2, 3, \cdots)$ having the form which results from that in (13) on replacing $n_1, \cdots, n_r$ each by ∞. Analogues of many of the additional foregoing results persist. In the formulas thus obtained we have a useful heuristic guide to certain expansion problems in the theory of difference equations, a part of which we shall presently indicate.

4. *Transcendental Expansion Problems.* Let us consider a fixed interval $a \leqq s \leqq b$ of the real s-axis, and let us associate with certain points of this interval the discrete values of i and j in equations (5) and (6). For the hth system in either (5) or (6), we take on (ab) a set of points n_h in number including the points a and b; and we interpolate x_{hj} into a function $x_h(s)$ by the method of linear interpolation described above. Similarly a_{khij} is interpolated into a function $a_{kh}(s, t)$. Equations (5) and (6) may now be looked upon as establishing relations among the functional values of these functions of s and t at certain points only of the axes of s and t; and so of defining the solution functions at these points and these alone, their definitions being completed by the methods of interpolation agreed upon.

To these systems we now apply certain limiting processes, allowing the numbers n_h (or at least a part of them) to increase indefinitely. As they increase, the functions $x_h(s)$ and $a_{kh}(s, t)$ pass through a corresponding sequence of changes. If the processes involved lead to replacing the original equations by well-defined limiting equations and their solutions by well-defined functions, we have in the process a suggestion of a heuristic guide to probable solutions of the limiting problems and to certain probable fundamental properties of these, together with intimations as to how they shall be established.

Usually we shall require that the distribution of basic points on the interval (ab) of the s-axis shall undergo change in such way that the norm of the distribution shall approach zero. By such processes one may realize, for instance, integral equations and differential equations as limiting cases. Among the characteristic results, for the finite case, which persist after certain limiting operations of this type have been performed are those relating to conjugacy, expansions, and representation of the latter by contour integrals. Without going into details, we may state the general limiting forms of certain equations, special cases of which we shall have occasion to consider. Equations (8), (12), (13) pass into the following limiting forms:

$$\int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \prod_{h=1}^{r} x_h^{(\rho)}(s_h) y_h^{(\sigma)}(t_h) ds_1 dt_1 \cdots ds_r dt_r \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{20}$$

$$z(s_1, s_2, \cdots, s_r) = \sum_{k=1}^{\infty} c_k x_1^{(k)}(s_1) x_2^{(k)}(s_2) \cdots x_r^{(k)}(s_r), \tag{21}$$

$$c_k = \frac{\int_a^b \int_a^b \cdots \int_a^b z(s_1, s_2, \cdots, s_r) D(s_1, t_1, \cdots, s_r, t_r) y_1^{(k)}(t_1)\, y_2^{(k)}(t_2) \cdots y_r^{(k)}(t_r) ds_1 dt_1 \cdots ds_r dt_r}{\int_a^b \int_a^b \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \prod_{h=1}^{r} x_h^{(k)}(s_h) y_h^{(k)}(t_h) ds_1 dt_1 \cdots ds_r dt_r}. \tag{22}$$

Corresponding to (10), (11), (14), (15) we have special cases of importance which indicate more clearly the essential simplicity of the formulas, namely,

$$\int_a^b \int_a^b b(t, s) x^{(\rho)}(s) y^{(\sigma)}(t) ds dt \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{23}$$

$$\int_a^b b(s) x^{(\rho)}(s) y^{(\sigma)}(s) ds \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{24}$$

$$z(s) = \sum_{k=1}^{\infty} c_k x^{(k)}(s), \qquad (a \leqq s \leqq b), \tag{25}$$

$$c_k = \frac{\int_a^b \int_a^b b(t, s) z(s) y^{(k)}(t) ds dt}{\int_a^b \int_a^b b(t, s) x^{(k)}(s) y^{(k)}(t) ds dt} \tag{26}$$

or

$$\frac{\int_a^b b(s) z(s) y^{(k)}(s) ds}{\int_a^b b(s) x^{(k)}(s) y^{(k)}(s) ds} .$$

If for the last form of c_k we have $c(s, t) = c(t, s)$, so that $y^{(k)}(s)$ may be taken equal to $x^{(k)}(s)$, and if $b(s) \equiv 1$, we have in (25) and the latter form of c_k in (26) formulas for the formal expansion of an "arbitrary" function of a single variable in terms of orthogonal functions of that variable; so that all the formulas (20) to (26) are generalizations of classic relations in the expansion of functions.

In these formulas the number of variables in the function z to be expanded is equal to the number of parameters involved in the problem. This correspondence is not essential. The desired extension can best be brought out by starting from the particular expansion (16). By linear interpolation we first obtain (18). Then we may proceed to the limiting case in such wise that μ and ν simultaneously approach infinity, the norm of the corresponding distributions of points on the s-axis and the t-axis approaching zero. We are thus led heuristically to an expansion of the form

$$z(s, t) = \sum_{k=1}^{\infty} c_k x^{(k)}(s, t), \tag{27}$$

where the x's are now solutions of the limiting problem. It is easy to see that the properties of conjugacy are maintained formally and that we may therefore readily determine the coefficients c_k. We may also proceed from (18) to the limit in another way, namely, by holding μ fixed and allowing ν to become infinite as before; we are thus led to expansions of the form

$$z_i(s) = \sum_{k=1}^{\infty} c_k x_i^{(k)}(s), \quad (i = 1, 2, \cdots, \mu), \tag{28}$$

for expanding a system of μ given functions in terms of μ sets of functions, the coefficients c_k of the expansion being the same for each of the given functions. This remarkable type of expansion I saw first in the manuscript of Dr. C. C. Camp's dissertation which he was kind enough to allow me to read; it occurred there in connection with a particular system of two linear differential equations of the first order.* In (27) we have an expansion of the type which arises in the theory of partial differential equations in two independent variables. In (28) we have the type of expansions which arises in the theory of n linear differential equations of the first order. It is clear that the type of extension employed in this paragraph for the case of one parameter may be utilized in a variety of ways in connection with problems involving r parameters.

Let us consider the results to which this heuristic guide leads us in the case of the adjoint differential systems

$$(29)\qquad \frac{dy_i}{dx} = \sum_{j=1}^{n} (a_{ij} + \lambda\alpha_{ij})y_j, \qquad (i = 1, 2, \cdots, n),$$

$$(30)\qquad \frac{dz_i}{dx} = \sum_{j=1}^{n} (- a_{ji} - \lambda\alpha_{ji})z_j, \qquad (i = 1, 2, \cdots, n).$$

If for fixed i we multiply these respectively by z_i and y_i, add the resulting equations member by member, sum as to i from 1 to n, and then integrate from a to b (a range in which the coefficients are assumed to be continuous) we have

$$(31)\qquad [y_1z_1 + y_2z_2 + \cdots + y_nz_n]_{x=a}^{x=b} = 0.$$

The first member of this relation is a non-singular bilinear form in the two sets of $2n$ variables each,

$$(32)\qquad y_1(a), y_2(a), \cdots, y_n(a), y_1(b), \cdots, y_n(b);$$
$$z_1(a), \cdots, z_n(a), z_1(b), \cdots, z_n(b).$$

It can be written in an infinite number of ways in the form

$$(33)\qquad [y_1z_1 + \cdots + y_nz_n]_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} Y_i(y)Z_i(z),$$

where $Y_i(y)[Z_i(z)]$ is a set of $2n$ linearly independent homo-

* Since this was written a paper by A. Schur has appeared dealing with this problem; see MATHEMATISCHE ANNALEN, vol. 82 (1921), pp. 213–236. It contains (p. 214) a reference to a special case of the problem treated by Hilbert (GÖTTINGER NACHRICHTEN, 1906, pp. 474–480). This I had not seen before.

geneous linear functions of the $2n$ variables $y[z]$ of (32). With (29) and (30), respectively, we associate the boundary conditions

$$(34) \qquad Y_i(y) = 0, \qquad (i = 1, 2, \cdots, n),$$

$$(35) \qquad Z_i(z) = 0, \qquad (i = n + 1, \cdots, 2n).$$

Then the characteristic values λ for problem (29), (34) are the same as those for problem (30), (35). We suppose that the conditions are set up in such a way that the number of these characteristic values is infinite. We denote the characteristic values and the corresponding solutions by $\lambda^{(k)}$, $y_i^{(k)}, z_i^{(k)}$, $(k = 1, 2, 3, \cdots)$. Corresponding to (20), (21), (22) we now have the relations

$$\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ji} y_i^{(k)} z_j^{(l)} \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f_i(x) = \sum_{k=1}^{\infty} c_k y_i^{(k)}, \qquad (i = 1, 2, \cdots, n),$$

$$c_k = \frac{\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ji} f_i z_j^{(k)} dx}{\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ji} y_i^{(k)} z_j^{(k)} dx}, \qquad (k = 1, 2, 3, \cdots).$$

If we apply to systems (29) and (30) a limiting process often employed by Volterra we are led to adjoint integro-differential equations of the form

$$\frac{\partial y(x, s)}{\partial x} = \int_\alpha^\beta \{u(x, s, t) + \lambda v(x, s, t)\} y(x, t) dt,$$

$$\frac{\partial z(x, s)}{\partial x} = \int_\alpha^\beta \{- u(x, t, s) - \lambda v(x, t, s)\} z(x, t) dt,$$

where the range of variation of x is from a to b while that of s and t is from α to β. Corresponding to the last three equations of the preceding paragraph we now have the following:

$$\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s)\, y_k(x, s)\, z_l(x, t)\, ds\, dt\, dx \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f(x, s) = \sum_{k=1}^{\infty} c_k y_k(x, s),$$

$$c_k = \frac{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s) f(x, s)\, z_k(x, t)\, ds\, dt\, dx}{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s)\, y_k(x, s)\, z_k(x, t)\, ds\, dt\, dx},$$

$$(k = 1, 2, \cdots),$$

where $y_k(x, s)$, $z_k(x, s)$ denote the solutions corresponding to the characteristic value λ_k of the set $\lambda_1, \lambda_2, \cdots$ of distinct characteristic values.

The same procedure may be applied with equal facility to the adjoint systems of integro-differential equations

$$\frac{\partial y_i(x, s)}{\partial x} = \sum_{j=1}^{n} (a_{ij} + \lambda \alpha_{ij})\, y_j(x, s) + \int_a^\beta \sum_{j=1}^{n} [\rho_{ij}(x, s, t) + \lambda \sigma_{ij}(x, s, t)]\, y_j(x, t)\, dt,$$

$$\frac{\partial z_i(x, s)}{\partial x} = \sum_{j=1}^{n} (- a_{ji} - \lambda \alpha_{ji})\, z_j(x, s) + \int_a^\beta \sum_{j=1}^{n} [- \rho_{ji}(x, t, s) - \lambda \sigma_{ji}(x, t, s)]\, z_j(x, t)\, dt,$$

for $i = 1, 2, \cdots, n$, and to various generalizations and extensions of them. In this way emerge formal properties of various types of expansions arising in connection with differential and integro-differential equations. The problems may likewise be set up with equal facility for the case of an equation or system of equations of any order with respect to differentiation instead of merely for the first order as in the foregoing problems. Furthermore, one can treat equally well a system with r parameters λ involved in a way analogous to that observed in connection with the algebraic problem defined at the beginning of § 3. The formulas necessarily become more complicated, but the fundamental guiding ideas are unmodified; the algebraic theory indicates the whole procedure and suggests the principal results as limiting cases of the algebraic propositions.

The way in which the corresponding problem may be set up for the difference equation will be indicated by a very brief statement. Let us consider the adjoint systems of difference equations

$$(36) \quad u_i(x + 1) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda \psi_{ij})\, u_j(x), \qquad (i = 1, 2, \cdots, n),$$

$$(37) \quad v_i(x) - v_i(x + 1) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda \psi_{ji})\, v_j(x + 1), \qquad (i = 1, 2, \cdots, n),$$

where φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and vanish there to an order at least as high as the second. We confine attention to those solutions alone which have the property that each function in a solution approaches a (finite) limiting value as x approaches infinity along any ray from the origin exclusive of the negative axis of imaginaries or along any line proceeding to the right parallel to the axis of reals.

If we multiply (36) through by $v_i(x+1)$ and (37) through by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n} \Delta\{u_i(x)v_i(x)\} = 0. \tag{38}$$

If the real part of a is sufficiently large we have u_i and v_i analytic at every point x whose real part is not less than the real part of a. Hence in (38) we may sum as to x from a to infinity, where x runs over the values $a, a+1, a+2, \cdots$; thus we have

$$\sum_{i=1}^{n} \{u_i(\infty)\, v_i(\infty) - u_i(a)\, v_i(a)\} = 0. \tag{39}$$

Let us now suppose that adjoint homogeneous linear boundary conditions, implying (39), are set up on the $u_i(\infty)$, $u_i(a)$ and on the $v_i(\infty)$, $v_i(a)$ similar to conditions (34) and (35) in a similar problem above and let us suppose that we have the infinite set of characteristic values and corresponding solutions $\lambda^{(k)}$, $u_i^{(k)}$, $v_i^{(k)}$, $(k = 1, 2, 3, \cdots)$. The fundamental formulas for the expansion problem thus arising are the following:

$$\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t)\, u_i^{(k)}(a+t)\, v_j^{(l)}(a+1+t) \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad (i = 1, 2, \cdots, n),$$

$$c_k = \frac{\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t) f_i(a+t)\, v_j^{(k)}(a+1+t)}{\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t)\, u_i^{(k)}(a+t)\, v_j^{(k)}(a+1+t)},$$

$$(k = 1, 2, 3, \cdots).$$

If a classic limiting process of Volterra is applied to systems (36) and (37), they go over into adjoint integro-difference equations analogous to the integro-differential equations treated above; and these may be generalized to related systems just as we generalized the corresponding problem for integro-differential equations. In all these cases the properties of conjugacy and the formal results for expansions persist in the form naturally to be expected. Moreover, corresponding and closely similar theories exist for q-difference and integro-q-difference equations.

In view of the basic algebraic theory and the transcendental problems already treated it is clear that there must exist in the theory of integral equations expansion problems involving r parameters not only in the classic case when $r = 1$ but also in the general case when r is any positive integer. Moreover, if we think of the several types of expansion problems—those for differential, difference, q-difference, integral, integro-differential, integro-difference, and integro-q-difference equations—in intimate connection with the basic algebraic theory, it becomes apparent that the case of r parameters (for $r > 1$) is not confined to a set of r equations of the same sort. There is nothing to prevent one subset of the basic algebraic equations from proceeding to differential equations as limiting forms, another to difference equations, another to integral equations, another to q-difference equations, and so on. Thus we can see beforehand that we may formulate the expansion problem for a variety of mixed systems. One is in fact led naturally to such systems in the consideration of certain integro-differential, integro-difference, and integro-q-difference equations. We shall not take space to treat any of these mixed systems, preferring rather to exhibit briefly the nature of the problem for partial differential equations. From these one may proceed naturally to related integro-differential equations. Similar problems may be formulated for partial difference and integro-difference equations.

Let us consider the adjoint partial differential equations

$$L(u) + \lambda L_1(u) = 0, \qquad M(v) + \lambda M_1(v) = 0,$$

where

$$L(u) \equiv a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u,$$

$$L_1(u) \equiv p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + ru,$$

the symbols $u, v, a, b, c, \alpha, \beta, \gamma, p, q, r$ denoting functions of the independent variables x and y continuous and suitably differentiable in the square $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$. It is easy to show the existence of an identity of the form

$$\int_0^1\int_0^1[v\{L(u)+\lambda L_1(u)\}-u\{M(v)+\lambda M_1(v)\}]dxdy \equiv \int_0^1 Bdt,$$

where B is a sort of bilinear form in the functions u and v and their first derivatives, the arguments of the functions being suitably restricted. If suitable boundary conditions are set up for the u-problem and for the v-problem so that $B \equiv 0$ in virtue of the boundary conditions and so that we have the infinite set of characteristic values and corresponding solutions $\lambda_k, u_k, v_k, (k = 1, 2, 3, \cdots)$, then we have the following fundamental formulas:

$$\int_0^1\int_0^1 u_i M_1(v_j)\, dx\, dy \begin{cases} = 0 \text{ if } i \neq j, \\ \neq 0 \text{ if } i = j, \end{cases}$$

$$f(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y),$$

$$c_k = \frac{\int_0^1\int_0^1 f(x, y)\, M_1(v_k)\, dx\, dy}{\int_0^1\int_0^1 u_k(x, y)\, M_1(v_k)\, dx\, dy}, \quad (k = 1, 2, 3, \cdots).$$

These results are readily carried over to partial differential equations of other forms and to the case of much more general regions than the square over which we have integrated in this particular instance.

5. *Transcendental Oscillation and Comparison Theorems.* The fundamental algebraic oscillation theorem in the earlier part of § 2 has several limiting forms of interest. We consider first those for homogeneous linear differential equations. The result is classic for equations of the second order: the zeros of two linearly independent solutions of such an equation separate each other throughout any interval containing no singular point of the equation. We shall now state one

extension of this result to equations of order n, $n > 2$. Such an equation we write in the general form

$$u^{(n)} + p_1 u^{(n-1)} + \cdots + p_{n-1}u' + p_n u = 0, \tag{40}$$

where the superscripts refer to differentiation and where the coefficients p are real-valued, single-valued, continuous functions of the real variable x on the interval $(a\ b)$. Since this involves an n-fold infinitude of solutions we shall require boundary conditions to restrict the permissible solutions to a two-fold infinitude linearly dependent on two linearly independent solutions (this being done so that the new theorem shall indeed be a direct limiting form of the algebraic theorem referred to). Suitable boundary conditions may be expressed by means of Stieltjes integrals in the form

$$\sum_{j=1}^{\nu} \int_a^b L_{ij}(u)\, d\psi_{ij}(x) = 0, \tag{41}$$

$$(i = 1, 2, \cdots, n-2;\ \nu = \text{positive integer}),$$

where the $\psi_{ij}(x)$ are functions of bounded variation on (ab) and the $L_{ij}(u)$ denote homogeneous linear expressions in u, u', $\cdots$, $u^{(n-1)}$. As a special case we have conditions which reduce to the following: $u(a) = 0, u'(a) = 0, \cdots, u^{(n-3)}(a) = 0$.

By aid of a fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ of solutions of (40) we define constants λ through the formulas

$$\sum_{k=1}^{\nu} \int_a^b L_{ik}(\bar{u}_j)\, d\psi_{ik}(x) = \lambda_{ij}, \qquad \begin{pmatrix} i = 1, 2, \cdots, n-2 \\ j = 1, 2, \cdots, n \end{pmatrix},$$

and then the determinant $D(x)$,

$$D(x) \equiv \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2,\,1} & \lambda_{n-2,\,2} & \cdots & \lambda_{n-2,\,n} \end{vmatrix}.$$

The zeros of $D(x)$ are independent of the choice of the fundamental system by means of which they are defined; that is, they depend only on (40) and (41). These zeros we shall call the *special points* of (ab) for the problem (40), (41). Then we have the following theorem:

On any interval of (ab) containing no special points for the *problem* (40), (41) *the zeros of any two linearly independent solutions of* (40), (41) *separate each other.*

The foregoing result may be applied to the case of any two linearly independent solutions of (40) without reference to any *preassigned* boundary conditions. For this purpose we associate with any two linearly independent solutions u_1, u_2 of (40) a set of boundary conditions capable of representation in the form (41) and having the property that the solutions of (40) and the determined conditions (41) are those functions and those alone which are linearly dependent upon u_1 and u_2. As a simple example of such associated boundary conditions we have those determined as follows: Let the initial constants for u_1 and u_2 at a point $x = \alpha$ of (ab) be

$$u_i^{(k)}(\alpha) = \rho_{ik}, \qquad (k = 0, 1, 2, \cdots, n-1;\ i = 1, 2).$$

Let the coefficients σ_{ij} be so chosen that the equations

$$\sigma_{i0}u(\alpha) + \sigma_{i1}u'(\alpha) + \sigma_{i2}u''(\alpha) + \cdots + \sigma_{i,\,n-1}u^{(n-1)}(\alpha) = 0, \qquad (i = 1, 2, \cdots, n-2),$$

have those solutions and those alone which may be written in the form $u^{(k)}(\alpha) = a_1\rho_{1k} + a_2\rho_{2k}$, $(k = 0, 1, \cdots, n-1)$, where a_1 and a_2 are arbitrary constants. Having thus introduced suitable boundary conditions restricted by means of the given solutions u_1 and u_2, we may define the determinant $D(x)$, and hence the special points, in the way indicated for the preceding case. By means of these points we may divide the interval (ab) into segments on the interior of each of which the zeros of u_1 and u_2 separate each other in accordance with the foregoing theorem.

As a second case, let us consider the difference equation

$$L(x)\,u(x) + M(x)\,u(x+1) + N(x)\,u(x+2) = 0$$

in which all the indicated functions are real-valued, single-valued, continuous functions of the real variable x for $x \geqq \alpha$, and $L(x)$ and $N(x)$ are both of one and the same sign for $x \geqq \alpha$. Let $u_1(x)$ and $u_2(x)$ be a fundamental system of solutions of this equation and let a $(a \geqq \alpha)$ be a point for which $w(a) \neq 0$ where $w(x) = u_1(x)\,u_2(x+1) - u_1(x+1)\,u_2(x)$. Let $\bar{u}_i(x)$ be the function obtained by linear interpolation from the set of constants $u_i(a)$, $u_i(a+1)$, $u_i(a+2)$, $\cdots$, with respect to a system of coordinate axes obtained by drawing lines perpendicular to the x-axis through the points a, $a+1$, $a+2$, $\cdots$. Let the zeros of $\bar{u}_i(x)$ on the range $a \leqq x < \infty$ be called the characteristic points of $u_i(x)$ with respect to a. Then we have the following theorem analogous to that of Sturm for a second order differential equation: *The characteristic points of $u_1(x)$ and $u_2(x)$ with respect to a separate each other.*

This result admits of extension to a system formed of a difference equation of order n, $n > 2$, and $n-2$ boundary conditions of a certain general sort restricting the simultaneous solutions to a two-fold infinitude. Similar results may also be obtained for q-difference equations. In fact, those for difference and q-difference equations are both special cases of like results for a rather general class of functional equations including difference and q-difference equations as special cases; but we shall not here take the space necessary to set forth these more general results. The comparison theorems which follow suggest their nature. The second theorem of § 2 has as a limiting case a theorem which is essentially equivalent to the following classic Sturmian theorem of comparison.

Let us consider the two differential equations

$$\frac{d}{dx}(K_1 u') - G_1 u = 0, \tag{42}$$

$$\frac{d}{dx}(K_2 u') - G_2(u) = 0, \tag{43}$$

in which K_1, K_2, G_1, G_2 are functions which are continuous throughout the interval (ab) defined by the inequalities $a \leqq x \leqq b$ and in this interval satisfy the relations

$$0 < K_2 \leqq K_1, \qquad G_2 \leqq G_1. \tag{44}$$

Moreover, let u_1 and u_2 be solutions of (42) and (43), respectively, neither of which is identically zero in (ab). Then if x_1 and x_2 are any two consecutive zeros of u_1 in (ab) there is at least one zero of u_2 in the interior of the interval (x_1x_2) provided either that at least one equality sign in (44) fails to hold at every point of the interval (x_1x_2), or that u_1 and u_2 are linearly independent in (x_1x_2).

The two results stated in the paragraphs following the theorem referred to in § 2 have as limiting cases two theorems which are also classic in the theory of Sturm. The algebraic theorems were indeed suggested by these Sturmian theorems. The latter, with considerable loss of elegance, have been extended to homogeneous linear differential equations of general order k (ANNALS OF MATHEMATICS (2), vol. 19 (1918), pp. 159–171). It is possible to extend the general algebraic results of the latter part of § 2 to the analogous algebraic case, namely, the case of algebraic systems with k linearly independent solutions; but the results lack (in some respects) the desired elegance. From them one may in turn obtain corresponding properties of a certain class of functional equations. We content ourselves with giving some of these results for the most interesting case, namely, that in which the equations have just two independent solutions.

Let us consider the substitution $x' = S_x$, denoting its nth power by $x' = S_x^n$. Let it be such that there exists an open interval I of the real x-axis, such that $\lim S_{x_0}^n = \beta$ for every x_0 of I, β being an end-point of I and the limit being approached monotonically. Then consider the functional equations

$$u(S_x^2) + \varphi(x)u(S_x) + u(x) = 0, \tag{45}$$

$$v(S_x^2) + \psi(x)v(S_x) + v(x) = 0, \tag{46}$$

in which $\varphi(x)$ and $\psi(x)$ are real-valued, single-valued, continuous functions of the real variable x on the interior of the interval I. Suppose, furthermore, that S_x is such that each of these equations has a fundamental system of solutions consisting of two functions which are real-valued, single-valued, and continuous on the open interval I. (In case $S_x \equiv x + 1$ and I is the interval $\alpha < x < \infty$, our equations are ordinary difference equations; in case $S_x = qx$, q being real and greater than unity, and I is the interval $0 \leqq \alpha < x < \infty$, our equations are q-difference-equations.) If a is an interior point of the interval I, we define the characteristic points of a function $t(x)$ with respect to a to be the zeros on the interval $a \leqq x < \beta$ (or $a \geqq x > \beta$) of the function $\bar{t}(x)$ derived from the constants $t(a)$, $t(S_a)$, $t(S_a^2)$, $\cdots$ by linear interpolation with respect to the system of coordinate axes obtained by drawing lines perpendicular to the x-axis through the points a, S_a, S_a^2, $\cdots$. Let u and v be real-valued, single-valued, continuous solutions of equations (45) and (46), respectively. Then we have the following three theorems:

If $u(x)$ has two consecutive characteristic points with respect to a on the μth and $(m+1)$th intervals $(\mu < m)$ of the set of intervals whose end-points are the consecutive pairs of the sequence a, S_a, S_a^2, S_a^3, $\cdots$, then $v(x)$ has a characteristic point between these characteristic points of $u(x)$ provided that either

(*a*) $\varphi(x) \leqq \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, the equality sign not holding for all the end-points of these intervals; or,

(*b*) $\varphi(x) = \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, and the two sets of constants

$$u(S_a^{\mu-1}),\ u(S_a^{\mu}),\ \cdots,\ u(S_a^{m-1}),$$
$$v(S_a^{\mu-1}),\ v(S_a^{\mu}),\ \cdots,\ v(S_a^{m-1}),$$

are linearly independent.

Next, let us suppose that $u(a) \neq 0$, $v(a) \neq 0$, $u(S_a)/u(a) > v(S_a)/v(a)$; and that $\varphi(x) \leqq \psi(x)$ for $x = a, S_a, S_a^2, \cdots, S_a^{\nu-1}$. If $u(x)$ has k characteristic points on the ν intervals whose end-points are the consecutive pairs of the sequence a, S_a, S_a^2, $\cdots$, S_a^{ν}, then $v(x)$ has at least k characteristic points on these intervals; and the jth of these characteristic points of $v(x)$ (counted from a towards S_a^{ν}) is nearer to a than the jth characteristic point of $u(x)$.

In the third place, let $u(a)$, $v(a)$, $u(S_a^k)$, $v(S_a^k)$ be all different from zero and let $u(S_a)/u(a) > v(S_a)/v(a)$. Let $u(x)$ and $v(x)$ have the same number (which may be zero) of characteristic points on the k intervals whose end-points are the consecutive pairs of the sequence a, S_a, S_a^2, $\cdots$, S_a^k. Then we have

$$\frac{u(S_a^{k+1})}{u(S_a^k)} > \frac{v(S_a^{k+1})}{v(S_a^k)},$$

provided that $\varphi(x) \leqq \psi(x)$ for $x = a, S_a, \cdots, S_a^{k-1}$.

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