BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

THE TWENTY-EIGHTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

THE twenty-eighth summer meeting of the Society was held at Wellesley College, Wellesley, Massachusetts, September 7–9, 1921, in conjunction with the meeting of the Mathematical Association of America. Joint sessions devoted to discussions of the theory of relativity were held by the two organizations on Wednesday afternoon and Thursday morning (September 7–8). The joint dinner on Wednesday evening was attended by one hundred eleven members and friends. At the close of the sessions, a resolution was passed thanking the college authorities and the Wellesley members of the committee on arrangements for their generous hospitality.

Tower Court was opened for the accommodation of the visitors. By courtesy of Mrs. Walter Hunnewell, the Hunnewell Gardens were open to visitors on Tuesday afternoon. On Tuesday evening an organ recital was given in Houghton Memorial Chapel by Professor H. C. Macdougall of Wellesley College, after which many embraced the opportunity of visiting Whitin Observatory. On Wednesday afternoon, the Denton Brothers' remarkable collection of butterflies was an attraction to many. On Thursday evening, the treasure room of the Library was open, and many ancient books and manuscripts, both mathematical and otherwise, were on exhibition. An automobile excursion to Concord and Lexington was held on Friday afternoon.

Besides the joint sessions, there were regular meetings for the presentation of papers on Thursday afternoon and on Friday morning. The attendance at all the scientific sessions included the following ninety-one members of the Society:

Archibald, Bacon, Barnett, Barney, S. R. Benedict, Bennett, Bill, Birkhoff, Blair, Bliss, Blodgett, Borden, E. W. Brown, Cairns, B. H. Camp, Chace, G. M. Conwell, Coolidge, Copeland, Crum, Currier, Dimick, Doak, Donohue, Durfee, Eiesland, Eisenhart, Ettlinger, Falconer, Foberg, Fry, Gillespie, Gilman, Glenn, Gummer, Haskins, Hedrick, Archibald Henderson, Horton, Howland, Hulburt, Huntington, Hurwitz, Dunham Jackson, Kellogg, W. D. Lambert, F. P. Lewis, Longley, Lunn, Mathews, H. A. Merrill, A. L. Miller, G. A. Miller, H. H. Mitchell, C. N. Moore, H. C. M. Morse, Musselman, Norwood, Osgood, Palmié, Pell, Pierpont, V. C. Poor, Rambo, C. N. Reynolds, Harris Rice, L. H. Rice, R. G. D. Richardson, Ritt, E. D. Roe, J. R. Roe, Rutledge, Safford, Simon, C. E. Smith, P. F. Smith, Tripp, Tyler, Vivian, J. L. Walsh, Webster, H. S. White, Whittemore, Wiener, R. G. Wood, F. S. Woods, Worthington, Yeaton, J. M. Young, J. W. Young, Zeldin.

President Bliss occupied the chair, being relieved by Vice-President Jackson and by Professor G. A. Miller.

The Council met in Tower Court on Thursday. The following were elected to membership in the Society:

Dr. Nina May Alderton, University of California;
Professor James Atkins Bullard, United States Naval Academy;
Professor Frank Hollinger Clutz, Pennsylvania College;
Dr. Paul H. Daus, University of California;
Mr. James Strode Elston, Travelers Insurance Company;
Mr. Laurence Monroe Klauber, San Diego Consolidated Gas and Electric Company;
Dr. Cyrus Colton MacDuffee, Princeton University;
Mr. Edward Bontecou Morris, Travelers Insurance Company;
Professor Henry Martyn Robert, United States Naval Academy;
President Levi Stephen Shively, Mount Morris College;
Dr. Daniel Victor Steed, University of California.

Thirty applications for membership were received.

The report of the Budget Committee indicated a deficit for the year 1921 and a larger one in prospect for 1922, unless prompt measures are taken to remedy the situation. While there has been an increase in receipts over 1916 of about four thousand dollars, the expenditures have mounted more than five thousand, due in large measure to the increase in the cost of printing. It was decided to inaugurate a campaign for new members, and to this end to enlist the interest and assistance of all members of the Society. In order to facilitate this campaign, By-Law I was suspended for the months November, 1921, to February, 1922, inclusive.

It was voted that the Mathematical Society of France be approached in regard to a reciprocity agreement concerning dues. Committees were appointed to deal with the following matters: life membership fees and funds; the desirability of publishing the Chicago Colloquium lectures; sale and acquisition of back numbers of the journals and other publications of the Society; arrangements for the annual meeting.

The following papers were read at the joint sessions with the Mathematical Association of America:

Some mathematical aspects of the theory of relativity, by Professor James Pierpont.

The place of the Einstein theory in theoretical physics, by Professor A. C. Lunn.

Titles and abstracts of the papers read at the regular sessions follow below. Professor Schwatt's first two papers were read by Professor Eisenhart and his third paper by Professor H. H. Mitchell. The papers of Professor Kasner, Miss Hausle, Professors Synge, Fischer, Kline, R. L. Moore, and Dodd, Dr. Gronwall, Professor Alexander, Professor Rowe, Dr. Wiener, Dr. Zeldin, and Professor Porter, were read by title.

1. Professor L. P. Eisenhart: *Einstein static fields which admit a continuous group G_2 of transformations into themselves.*

For static phenomena in the Einstein theory the linear element of the space-time continuum can be taken in the form $ds^2 = V^2dx_0^2 - ds_0^2$, where $ds_0^2 = \Sigma a_{ik}dx_idx_k$ for $i, k = 1, 2, 3$ is the linear element of the physical space S; and V_0 and the functions a_{ik} $(= a_{ki})$ are independent of x_0, the coordinate of time. Bianchi has found that there are two and only two types of three-spaces S that admit a continuous group G_2 of transformations into themselves. The author proposed the problem of finding the physical spaces, referred to above, which admit such transformations, with the idea that the question of symmetry of an Einstein space should be looked upon in this manner. The problem has been solved in this paper with the result that there are two classes of spaces of the first type of Bianchi, and four of the second type. In all cases the quantities a_{ik} are expressible as algebraic functions of one or two of the coordinates x_i $(i \neq 0)$. The spaces found are, in general, different from any previously known, as is shown by their geometric properties.

2. Professor John Eiesland: *On the class of a certain type of Einstein spreads.*

The author investigates the class of an Einstein spread having the following line element:

(1) $\quad -ds^2 = (1+\varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi_1 dt^2,$

in which R, θ, φ are spherical coordinates and φ_1, φ_2 are arbitrary functions of R. The following theorems are proved:

1. If the space (1) is of the first class, the condition

$$\varphi_1'' = \frac{1}{2}\left[\frac{\varphi_1'\varphi_2'}{\varphi_2} + \frac{(\varphi_1')^2}{\varphi_1}\right] \tag{2}$$

must be satisfied, and conversely.

2. A space (1) is at most of the second class.

A representation of the space in a flat 5-space is obtained for the case that (2) is satisfied, and another representation in a flat 6-space when (2) is not satisfied. With a slight modification of the condition (2), these theorems also hold for the more general stationary space

$-ds^2 = (1+\varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi_3 dRdt - \varphi_1 dt^2.$

Applications are made to various spaces in the general theory.

3. Professor Edward Kasner: *The solar gravitational field and certain other fields completely determined by light rays.*

In this paper, which has been submitted for publication to the MATHEMATISCHE ANNALEN, it is shown that the solar field, taken in the Schwarzschild form, is exactly determined by its light rays.

In a paper (AMERICAN JOURNAL, vol. 43, Jan., 1921, p. 20) the proof was given only for approximately euclidean forms. The present proof is valid for the complete equations. The same general method is then applied to the Minkowski form.

Finally, the discussion is carried out for a very simple class of Einstein fields, namely, those in which the coefficients of the squares of the differentials are all functions of one variable, say x_1 (see this BULLETIN, vol. 27, p. 62). It is shown that such fields are exactly determined by their light properties, and therefore the orbits (geodesics) can be calculated if we know merely the light rays (nullgeodesics).

These fields may be represented as four-spreads imbedded in a seven-flat, in contrast to the solar field which the author has shown can be imbedded in a six-flat (AMERICAN JOURNAL, vol. 43, April, 1921, p. 130).

A corresponding discussion of dimensionality is carried out for Einstein's more general cosmological equations

$$G_{ik} - \tfrac{1}{4}g_{ik}G = 0.$$

4. Professor G. A. Miller: *Prime-power groups containing only one invariant subgroup of every index which exceeds this prime number.*

Among the theorems proved are the following: If a non-cyclic group of order p^m, p being a prime number, contains only one invariant subgroup of every index which exceeds p, it must contain a cyclic subgroup of index p when $p = 2$, but when $p > 2$, each of its invariant subgroups whose order exceeds p must be non-cyclic. In all cases such a group must contain a subgroup of index p which includes all its operators whose orders exceed p^2. When $m > p + 1$ there are three and only three non-cyclic groups of order p^m which have the property that each of them contains only one invariant subgroup of every index which exceeds p and one abelian subgroup of index p. If H_α is such an invariant subgroup of order p^α then the pth power of every operator in H_α, $m > \alpha > 0$, is found in $H_{\alpha-p+1}$, where $\alpha - p + 1$ is to be replaced by zero whenever it is a negative number or zero.

5. Professor G. D. Birkhoff: *General mean-value relations.*

In this paper the following general problem is treated: a function $f(x)$ and its derivatives are connected by one or more polynomial equations in which x has values $x_1, x_2, \cdots, x_n$; to determine when a specified linear differential expression in f necessarily vanishes within the interval of the x_i's.

6. Professor G. D. Birkhoff: *On plates of variable thickness.*

This paper contains a general method for the development of the elastic properties of plates of variable thickness. Only the case of constant thickness has been hitherto considered.

7. Professor E. V. Huntington: *Application of least squares to the problem of apportionment.*

The problem of apportionment may be attacked in two ways: (1) by the comparison plan, which seeks to minimize the inequality between each state and every other state; and

(2) by the summation plan, which seeks to minimize the total error of the apportionment. The comparison plan, as shown by the writer in earlier papers (presented December, 1920, and February, 1921), leads to only five distinct methods, among which the choice of the *method of equal proportions* appears to be clearly indicated. (See AMERICAN STATISTICAL QUARTERLY, Sept. 1921.) The summation plan, on the other hand, which was used by Professor Owens in his papers at the February and April meetings, is here shown to be unsatisfactory, since it leads to so many different methods that it would be difficult to make a convincing choice between them. It may be noted that the only methods which appear under both plans are the method of equal proportions and the Willcox-Owens method of major fractions. The paper concludes with a *rank-list* of all the seventy-two known methods arranged in the order in which they favor the larger states. In this list, the method of equal proportions occupies, in a genuine sense, the middle position.

8. Professor I. J. Schwatt: *The summation of series by means of generating functions.*

This paper gives a method for finding the sums of certain series, such as

$$\sum_{k=0}^{n}\binom{2n-k}{k}m^k, \qquad \sum_{k=0}^{n}k^p, \qquad \sum_{k=0}^{[p/2]}\binom{n}{k}\binom{n-k}{p-2k}2^{p-2k},$$

which the author was unable to obtain by any other method.

9. Professor I. J. Schwatt: *The expansion of any power of a multinomial.*

The author obtains a series expansion, $\Sigma_{n=0}^{\infty}(x^n/n!)P_n$, for any power of a polynomial, $y = (\Sigma_{m=0}^{h}a_m x^m)^p$, where p is any real number. The coefficients P_n are numbers easily obtained from an expression for the nth derivative of y. The special case of p a positive integer is also considered.

10. Professor I. J. Schwatt: *The operator* $[r(d/dr)]$ *on* $F(r)$.

It is shown that for any analytic function $F(r)$

$$\left(r\frac{d}{dr}\right)^p F(r) = \sum_{n=1}^{p}\frac{(-1)^n}{n!}\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^p r^n\frac{d^n}{dr^n}F(r).$$

Several applications are given.

11. Miss Eugenie C. Hausle (introduced by Professor Edward Kasner): *Geometric characterization of special singly infinite families of heat curves.*

The author has considered the geometric characterization of the singly infinite families of curves derived from certain limitations on the Fourier equation of heat. It is known that a system of curves $\varphi(x, y, t) = T$, where t denotes the time and T the temperature, obeys the Fourier equation. If the heat is required to be in equilibrium, $\partial\phi/\partial t = 0$. Let this be Case I, the isothermal case. Case II is obtained by putting $\partial\phi/\partial t = 1$ and Case III by putting $\partial\phi/\partial t = \varphi$.

Consider Case I. The equation of the given family of curves $\varphi(x, y) = c$ and Laplace's equation must be satisfied simultaneously. Then $f = dy/dx$ satisfies Laplace's equation. If γ and $\bar{\gamma}$ denote the curvatures of the given curves and their orthogonal trajectories, the preceding result gives (1) $\gamma_s + \bar{\gamma}_{\bar{s}} = 0$ as the geometric property, where $\gamma_s = dr/ds$.

In Case II, two relations are obtained, each of the fourth order. In Case III, two relations are obtained, one of which is identical with one of those in Case II. These relations completely characterize all ∞^1 systems of heat curves.

12. Professor J. L. Synge: *On the stability of a bicycle with a light frame.*

The bicycle under discussion consists of two similar wheels —solids of revolution with equatorial planes of symmetry— and two weightless frame pieces. The stability is discussed in the case in which the system has uniform rectilinear motion, the wheels rolling on a rough horizontal plane. Use is made of Routh's conditions that no zero of a polynomial should have its real part positive. The conditions for stability differ with the detail of the specification, and five results are given. Of these, two are of especial interest, viz. (1) that for certain specifications the motion is stable for speeds up to, but not exceeding, a certain limit; (2) that when the common axis of the frame pieces is vertical in steady motion, there is instability at all speeds. Definite examples are given.

13. Professor C. A. Fischer: *Note on the definition of a linear functional.*

A linear functional has usually been defined as one which is distributive and continuous, but a continuous functional has been defined in at least two ways which are not equivalent.

F. Riesz, Fréchet and others have defined a continuous functional as one which satisfies the equation

$$\lim_{n\to\infty} L(u_n(x)) = L(u(x)),$$

when the sequence of u_n's approaches $u(x)$ uniformly. More recently Lévy and W. L. Hart have required only that it converge to $u(x)$ in the mean. F. Riesz has proved that if L is distributive and the above equation is satisfied for uniformly convergent sequences, the functional is equivalent to a Stieltjes integral, such as $\int_a^b u(x)d\alpha(x)$.

In the present note it is proved that if the same equation is satisfied when the u_n's are required only to converge in the mean, the functional is equivalent to a Lebesgue integral such as $\int_a^b u(x)\beta(x)dx$.

14. Professor J. R. Kline: *Certain theorems concerning simple closed and open curves.*

In order that a closed connected set M should be a simple closed curve, the author shows that it is necessary and sufficient that M remain connected upon the removal of any proper connected subset. Thus there is no unbounded closed connected set M which is such that $M - g$ is connected for every proper connected subset g of M. However, if M is an unbounded closed connected set that remains connected upon the removal of any *unbounded* proper connected subset, then M must be either an open curve, a ray of an open curve, or the point set composed of a simple closed curve J and a ray of an open curve starting from P such that the ray has P and only P in common with J.

15. Professor J. R. Kline: *A theorem concerning connected sets which become totally disconnected upon the removal of a single point.*

An example was recently given in the FUNDAMENTA MATHEMATICAE of a connected point set M which contains a point P such that $M - P$ is totally disconnected, i.e. $M - P$ contains no connected subset consisting of more than a single point. Professor Kline shows that no connected set M can have more than one point P such that $M - P$ is totally disconnected. Indeed, if a connected set M has one point P such that $M - P$ is totally disconnected, then, if Q is any point of M different from P, $M - Q$ is connected.

16. Professor R. L. Moore: *Concerning connectedness im kleinen and a related property.*

Sierpinski has recently* shown that in order that a closed connected point set M should be a continuous curve it is necessary and sufficient that, for every positive number ϵ, M should be the sum of a finite number of *closed,* connected point sets each of diameter less than ϵ.

A point set will be said to have property S if, for every positive number ϵ, it is the sum of a finite number of connected point sets, each of diameter less than ϵ. Professor Moore shows that, as applied to bounded connected point sets, condition S is stronger than that of connectedness im kleinen† but weaker than that of *uniform* connectedness im kleinen. He shows also that in order that a simply connected bounded domain should have a continuous curve for its boundary, it is necessary and sufficient that it should have property S. Finally, it is also shown that if a bounded point set M is uniformly connected im kleinen, then the point set composed of M plus its boundary is connected im kleinen.

17. Professor E. L. Dodd: *The probability function for the sum of certain functions, with applications to the theory of errors.*

This paper extends a theorem of von Mises,‡ who, by the use of the Stieltjes integral, treats together continuous and discontinuous probability. The theorem for the sum of variables is extended to the sum of functions, each limited in a finite interval, and having a limited number of oscillations therein, and with moments absolutely convergent.

The general mean of n measurements m_k is defined as $g[\Sigma c_k f_k(m_k) \div \Sigma c_k]$ where the c_k are positive constants, the f_k are continuous increasing functions, and g is the inverse of $\Sigma c_k f_k(\xi) \div \Sigma c_k$. If each $f_k = f$, and $0 < c \leqq c_k \leqq d$, and the errors are small in comparison with the measurements, the precision of this mean increases as the square root of n; but a constant error will appear that is sometimes negligible.

18. Dr. T. H. Gronwall: *On power series with positive real part in the unit circle.*

Carathéodory has given the necessary and sufficient conditions on the n constants $a_1, a_2, \cdots, a_n$ in order that there shall

* Fundamenta Mathematicæ, vol. 1 (1920), pp. 44–60.

† Cf. H. Hahn.

‡ *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 4, pp. 24, 54, 61, 80, 81.

exist a $\varphi(z)$, holomorphic and of positive real part for $|z| < 1$, the expansion of which begins with

$$\varphi(z) = \tfrac{1}{2} + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots.$$

Using the methods of Minkowski's geometry of convex solids, he expresses the a's in parametric form, and through the algebra of Hermitian forms, Toeplitz and Fischer have transformed these conditions into algebraic inequalities involving $a_1, a_2, \cdots, a_n$ and their conjugates. In the present paper the results of these authors are proved by the most elementary function theoretic means, consisting mainly in the combination of complete induction with Schwarz's lemma.

19. Professor J. W. Alexander: *Some theorems on transformations with invariant points.*

This paper gives a new proof of an extended form of Brouwer's theorem on the transformation of an n-sphere into itself, and a number of applications of the generalized theorem.

20. Professor J. W. Alexander: *Theorem on the interior of a simply connected closed surface in three-space.*

In this paper it is proved that if S be a simply connected closed surface in three-space the interior and boundary of S may be mapped uniformly on the interior and boundary of a sphere. The proof can be generalized to n dimensions.

21. Dr. H. M. Morse: *A fundamental class of geodesics on closed surfaces of genus greater than unity.*

Dr. Morse in this paper considers geodesics, termed geodesics of class A, every segment of which, however extended, is at least as short as any other curve joining the end points of the given segment. The fundamental theorem of this paper is that corresponding to any semicircle lying in the upper half plane and with end points on the axis of reals, there exists on the given surface a geodesic of class A possessing an image on the half plane which if extended indefinitely in either sense approaches respectively the two end points of the semicircle, and such that the semicircle and image of the geodesic can be traced out simultaneously by two moving points, capable of being joined by a continuously moving curve whose image on the given surface remains less in length than some fixed constant. Conversely, corresponding to every geodesic of class A there is associated in the above manner some semicircle with end points on the axis of reals.

22. Professor A. G. Webster: *On the problem of steering an automobile around a corner.*

In order to avoid skidding, there must be an upper limit to the centrifugal force. If we seek to make the mean square of this a minimum, we find that the curve should be the elastica. The kinematics of steering is examined, as are the results of the hypothesis that in driving at constant speed the wheel is put over at constant velocity. The intrinsic equation of the curve is found, and in practical cases it, like the elastica, differs little from the Cornu spiral used in optics. In practice all these curves (that is for the first half, which is symmetrically repeated) may be well represented by a cubical parabola. Drawings are given, and safe speeds are stated.

23. Professor A. G. Webster: *On the principles of mechanical integrators for differential equations, especially those of exterior ballistics.*

The usual forms of integrator involve the revolving plate and wheel at various distances from the center, or the steered wheel which rolls in the direction of the integrated curve. To this the writer proposes to add a third device, like an automobile with the front wheels at right angles to the wheel-base, the length of which varies. Such a device enables one to draw involutes. In the proposed machine, two integrators are combined, one of which rides on top of the other. One describes the hodograph, the other the evolute of the trajectory and the trajectory itself. Templets or cams insert once for all the law of resistance of the air and the law of density-change with altitude. The trajectory is drawn at one operation, although the equations are of the second order.

24. Professor C. N. Moore: *On the Fourier's series of non-integrable functions.*

An integral over a finite range which fails to converge on account of the nature of the infinite discontinuities of the integrand may be summable by methods analogous to those used for divergent series and divergent integrals over an infinite range. In this way Fourier's series may be obtained which correspond to functions that are non-integrable in the ordinary sense. The principal result of this paper is the theorem that if the integral of a function is summable (Cr) for any $r > 0$, the corresponding Fourier's series will be summable (C1) to the value $\lim_{h\to 0} [\frac{1}{2}\{f(x+h) + f(x-h)\}]$ at every point for which this limit exists.

25. Professor C. F. Gummer: *A generalization of Laguerre's rule of signs.*

It was shown by Laguerre that, if $c_1, c_2, \cdots, c_n$ are any real numbers and if $p_1, p_2, \cdots, p_n$ are a descending sequence of rational numbers, the sum of the multiplicities of those roots of the equation $\Sigma_i c_i x^{p_i} = 0$ that exceed unity is not greater than the number of variations of sign in the following sequence: $c_1, c_1 + c_2, c_1 + c_2 + c_3, \cdots, c_1 + \cdots + c_n$. In this paper a proof is given that allows the p's to be irrational. The theorem is applied to some special types of equation.

26. Professor R. E. Gilman: *The functions analogous to Lebesgue constants for a series of Hermite polynomials.*

To each member of the class of functions absolutely integrable in the interval $-\infty$ to $+\infty$ and such that $|f(x)| \leqq 1$ there corresponds a formal expansion in Hermite polynomials, which may or may not be convergent. If the sum of the first n terms of this expansion be denoted by $s(x, n)$, there is a function of the given class which, for a fixed n and for a fixed $x = x_0$ on the given interval, makes $s(x_0, n)$ a maximum. The author determines the order of magnitude with respect to n of ρ_n, where ρ_n is the maximum of $s(x_0, n)$.

27. Professor O. E. Glenn: *Theory of invariant elements.*

Many theories tend toward the unity exemplified in general analysis. The theory of invariant elements, which is the foundation of the methods of the present paper, has furnished the fundamental viewpoint in papers by the writer on algebraic, modular, differential, and special algebraic invariant theories, particularly for binary quantics. The theory here developed is a synthesis based upon these with extension to n variables under a form as thoroughly definitive as seems feasible.

28. Dr. J. L. Walsh: *On the location of the roots of the jacobian of two binary forms.*

This paper presents the following result: Let $g(z)$ denote a polynomial whose roots of respective multiplicities $m_1, m_2, \cdots, m_n$ lie at the collinear points $\alpha_1, \alpha_2, \cdots, \alpha_n$, and let $\alpha_1', \alpha_2', \cdots, \alpha_n'$ denote the roots, whose multiplicities are $m_1', m_2', \cdots, m_n'$, of the derivative $g'(z)$. Suppose the loci of m_i roots of a polynomial $f(z)$ to be the interior and boundary of a circle C_i whose center is α_i and radius r $(i = 1, 2, \cdots, n)$, and that $f(z)$ has no other roots. Then the loci of the roots of $f'(z)$ are the interiors and boundaries of circles C_i' of centers α_1', α_2', $\cdots$, α_n' and common radius r. A circle C_i' exterior to all other circles C_j' contains m_i' roots of $f'(z)$.

This result can be extended to a number of circles C_i that have a common external center of similitude, to the derivatives of $f(z)$ of all orders, and to the jacobian of two binary forms.

29. Professor J. E. Rowe: *The power of a modern gun and of thunder.*

The first part of the paper is devoted to the comparison of the power at which a large gun works during the time of actual performance, and the power of a man. The second part is devoted to the explanation of a method by which the magnitude of the air disturbance in thunder may be calculated.

30. Professor J. R. Musselman: *Spurious correlation applied to urn schemata.*

In games of pure chance such as balls drawn from an urn, or coin-tossing, or dice-throwing some scheme must always be introduced to obtain correlation between the two sets of drawings, and hence the correlation arising is "spurious" correlation. The author applies the idea of spurious correlation to various urn schemata and obtains some interesting theorems. This method gives a simple proof for that urn schema of Rietz which has t/s for its correlation coefficient.

31. Professor W. L. Crum: *The significance of the partial correlation coefficient in the comparison of ordered statistical series possessing rectilinear trends.*

It is well known that, if two ordered statistical series, for instance two historical economic series, are subject to rectilinear trends, their correlation coefficient is influenced by that fact. In order to determine the degree of correlation which is independent of the normal trends, it is customary to *correct* the original items by subtracting the corresponding ordinates of their respective lines of trend. This paper shows that, when there is no lag in the correlation, the coefficient of correlation between the residuals obtained by the above process of correction is precisely the partial correlation coefficient between the two given series. In case there is lag, it is shown that the partial correlation coefficient must be altered by the introduction of a multiplier which is the correlation between the two time series, one of which is displaced relative to the other by an amount equal to the lag.

32. Professor W. L. Crum: *A tentative substitute for the standard deviation in the examination of the dispersion of an ordered statistical series.*

A statistical series may be assigned to one of two broad classes according as it consists merely of a list of numbers of independent arrangement, or has its items ordered relative to a particular variable. Typical of the first class are the series composed of experimental measurements, and the chief illustrations of the second class are to be found in the historical series in the field of economic statistics. The paper points out that the standard deviation fails to give full information about the dispersion of series of the second class, and, in particular, takes no account of a phase of the dispersion which may be called the rate of fluctuation. With a view to bringing out certain facts about this rate, a study is made of the coefficient of correlation between the items of the given series and the values of the variable relative to which these items are ordered, and it is shown that this coefficient is not an adequate measure of the rate. Finally the square root of the mean squared second order difference is set up tentatively as a general measure of the fluctuation-rate for ordered statistical series.

33. Professor B. H. Camp: *The value of a sample. Second paper.*

The fundamental theorem of the paper is this: Let $f(x)$ be any frequency distribution whatever, and let y be equal to the mean of k x's drawn, with replacements, from $f(x)$. The rth moment of the distribution of the y's approaches as a limit the value of the rth moment of the Gaussian curve as k becomes infinite.

The more important applications of this theorem center round a method of approximating the difference between the rth moment and its limit in special cases. This enables one to use advantageously a generalization of Tchebycheff's criterion, discovered first by Pearson, for finding the probability that a random mean will differ from its ideal value by as much as a prescribed amount.

Taken in conjunction with another theorem announced in the author's earlier paper, it furnishes a method of obtaining high moments of a point binomial. These are valuable in the theory of simple sampling, but the labor of computing them has been prohibitive.

34. Dr. Norbert Wiener: *A form of series for potential problems.*

This paper develops a generalization of the notion of sets of normal and orthogonal functions by substituting the rule of combination

$$f(x, y, z) \,|\, g(x, y, z) = \iiint_V \Delta f \Delta g d\tau + \iint_S f g d\sigma,$$

where V is a region of three-space and S its boundary, for the rule of combination $\int_a^b u(x)v(x)dx$ used in the definition of ordinary orthogonality. Application of these generalized normalized sets is made to the solution of Dirichlet's problem and similar problems concerning Poisson's equation.

35. Dr. S. D. Zeldin: *Some hydrodynamic aspects of group theory.*

It is known that the steady motion of a fluid can be represented by a one-parameter continuous group in three variables. By finding the functions f which admit the infinitesimal transformation

$$Kf = u(x, y, z)\frac{\partial f}{\partial x} + v(x, y, z)\frac{\partial f}{\partial y} + w(x, y, z)\frac{\partial f}{\partial z},$$

the lines of flow can be determined. This paper deals with the invariant configurations which arise by finding the points satisfying equations $u = 0, v = 0, w = 0$. The groups particularly considered are of projective and linearoid types.

36. Professor M. B. Porter: *Two-way series for Lebesgue integrals.*

The author proposes to define, *ab initio*, the integral of any positive summable function in n variables by a two-way series. Most of the essential theorems of Lebesgue's theory then follow from the density theorem, which is an immediate corollary of Vitali's covering theorem. In case the integrand is not of one sign, the form of the series is altered. This leads to the definition of integrals for *non*-summable functions, among others Harnack integrals and Cauchy principal-value integrals.

This paper will appear in a later number of this BULLETIN.

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Secretary.