ON THE GENERALIZATION OF CERTAIN FUNDAMENTAL FORMULAS OF THE MATHEMATICAL THEORY OF FINANCE.

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1. *Introduction.* Certain of the fundamental formulas in the mathematical theory of finance lack a generality which would add much to the usefulness of such formulas. The lack of generality is due to an assumption that the periodic returns from capital profitably invested will be invested at the same rate of interest at which the principal itself is invested. For example, the most fundamental formula in the mathematical theory of finance,

$$S = P(1 + i)^n,$$

is employed to give the amount to which a principal P will accumulate at the end of n years at compound interest at rate i. But it is assumed in its derivation that every payment of interest upon the principal will be immediately invested, that all the interest upon this interest will be immediately invested, that all the interest on this third system of interest will be immediately invested, and so on, all at the original rate of interest. A much more general formula would be obtained both in this case and in many other cases if an allowance were made for the fact that the interest upon the principal may be invested at another rate of interest.

The purpose of this paper is to derive various fundamental formulas based upon an assumption that the periodic returns from an investment will be invested at another rate of interest. A few formulas will also be given which assume that even the interest upon the investment of the regular interest payments, and all further interest income, will be invested at a third rate of interest, such as that of a savings bank. Such formulas will, of course, include as special cases the formulas ordinarily employed and even the cases where the interest payments are not invested at all.

2. *The Amount of an Investment of a Single Sum. Two Rates of Interest.* If P dollars are placed at interest at rate i for n years the annual payments of interest will constitute an

annuity of iP per year. If we assume that these payments and all further interest will be immediately invested at a rate r the total accumulations or *amount* at the end of n years will evidently be

$$S = P(1 + is_{\overline{n}|}), \tag{1}$$

to be valued at rate r. If it should prove possible to invest all interest payments at the rate of interest at which the original principal was invested, that is if $r = i$, the formula (1) reduces at once to the usual form $S = P(1 + i)^n$.

If the regular interest payments are not invested at all, that is if $r = 0$, formula (1) reduces to the form $S = P(1 + ni)$, which is the formula employed in simple interest.

If the regular interest payments at rate i were made m times a year and the interest at rate r upon these interest payments were paid and compounded t times a year the formula for the amount at the end of n years would be

$$S = P(1 + js_{\overline{n}|}^{(m)}), \tag{2}$$

to be valued at rate r/t, where j is the nominal rate of interest realized upon the original principal, and $s_{\overline{n}|}^{(m)}$ is the symbol generally adopted for the conventional amount of an annuity of 1 per annum but payable m times a year, that is,

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{\left(1 + \frac{r}{t}\right)^{tn} - 1}{\left(1 + \frac{r}{t}\right)^{t/m} - 1},$$

at rate r/t, or

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{s_{\overline{tn}|}}{s_{\overline{t/m}|}}, \tag{A}$$

at any rate r/t. In case t/m should prove to be fractional, the value of $1/s_{\overline{t/m}|}$ could be found in a table of values of $1/s_{\overline{n}|}$ for fractional values of n or in a table of values of $i/j_{(p)}$, where

$$\frac{1}{s_{\overline{1/p}|}} = p \cdot \frac{i}{j_{(p)}}.$$

If it were found possible to invest all the interest payments at the same rate of interest and at the same frequency of conversion at which the original principal was invested, that is if $r = j$ and $t = m$, the formula (2) would reduce to the form

$$S = P(1 + j/m)^{mn},$$

which is the formula ordinarily employed in compound interest expressed in terms of a nominal rate of interest payable several times a year.

3. *The Investment of a Single Sum. Three Rates of Interest.* It is certainly possible for the regular interest payments to be sufficiently large to enable the investor to invest them at a more favorable rate than he could realize at his savings bank, in which case he would be interested in the *amount* of his investment where three rates of interest would be involved. If the rate of interest realized on the original principal is j payable m times a year, the rate on these interest payments is r payable t times a year, and the rate on the second and all successive systems of interest payments is ρ payable θ times a year the amount at the end of n years, according to formula (2), is

$$S = P\{1 + \Sigma(j/m)(1 + rs_{\overline{x}|}{}^{(t)})\},$$

at rate ρ/θ, where the summation extends from $x = 0$ by intervals of $1/m$ to $x = n - 1/m$ inclusive. This expression reduces without difficulty to

$$(3) \qquad S = P\left\{1 + nj + \frac{jr\theta(s_{\overline{\theta/n}|} - mns_{\overline{\theta/m}|})}{mt\rho\, s_{\overline{\theta/m}|}\, s_{\overline{\theta/t}|}}\right\}.$$

It is easily verified that when $\rho = r$ and $\theta = t$ formula (3) reduces to formula (2).

4. *The Amount of an Annuity. Two Rates of Interest.* We shall consider first the amount or total accumulations of the simplest form of an annuity of 1 per annum. As the amount of an annuity is simply the sum of the amounts of all of the various payments, the amount of the annuity, which is denoted by S_n, is

$$S_n = (1 + is_{\overline{n-1}|}) + (1 + is_{\overline{n-2}|}) + \cdots + (1 + is_{\overline{1}|}) + 1$$
$$= n + i(s_{\overline{1}|} + s_{\overline{2}|} + \cdots + s_{\overline{n-1}|}),$$

at rate r, which reduces easily to

$$(4) \qquad S_n = n + (i/r)(s_{\overline{n}|} - n)$$

at rate r. If $r = i$, formula (4) evidently reduces at once to $s_{\overline{n}|}$.

If the annuity of 1 per annum were payable p times a year, the interest at rate j on these payments were payable m times a year, and the interest at rate r on these interest payments were payable and compounded t times a year the amount of the annuity would be, by (2),

$$S_n = \Sigma(1/p)(1 + js_{\overline{x}|}{}^{(m)}),$$

at rate r/t, where the summation extends from $x = 0$ by intervals of $1/p$ to $x = n - 1/p$ inclusive. This expression reduces without difficulty to

$$(5) \qquad S_n = n + \frac{jt(s_{\overline{tn}|} - nps_{\overline{t/p}|})}{pmr\, s_{\overline{t/m}|}\, s_{\overline{t/p}|}}.$$

at rate r/t.

If the annuity and all the interest payments were payable only once a year, that is if $p = m = t = 1$, formula (5) would reduce to formula (4). Similarly, if $j = r$ and $m = t$, formula (5) reduces to formula (A).

5. *The Amount of an Annuity. Three Rates of Interest.* If the annuity of 1 per annum were payable p times a year, the interest at rate j on these payments were payable m times a year, the interest at rate r on these interest payments were payable t times a year and the interest (ρ) on the second and all successive systems of interest payments were payable θ times a year the amount of the annuity, according to formula (3), would be

$$S_n = \Sigma \frac{1}{p}\left\{ 1 + xj + \frac{jr\theta(S_{\overline{\theta x}|} - mxS_{\overline{\theta/m}|})}{mtp\, s_{\overline{\theta/m}|}\, s_{\overline{\theta/t}|}} \right\},$$

at rate ρ/θ, where the summation extends from $x = 0$ by intervals of $1/p$ to $x = n - 1/p$ inclusive. This expression reduces then to

$$(6) \quad S_n = n + \frac{n\,(np-1)j}{2p}\left(1 - \frac{r\theta}{tps_{\overline{\theta/t}|}}\right) + \frac{jr\theta^2(s_{\overline{\theta n}|} - nps_{\overline{\theta/p}|})}{pmtp^2 s_{\overline{\theta/m}|} s_{\overline{\theta/t}|} s_{\overline{\theta/p}|}},$$

at rate ρ/θ.

It is easily verified that if $\rho = r$ and $\theta = t$, formula (6) reduces to formula (5). If $\rho = r = j$ and $\theta = t = m$, the formula reduces to the familiar $s_n{}^{(p)}$ at rate j/m.

6. *Present Values. Perpetuities.* The general formula for the present value of a single sum due at a future time where two or more rates of interest are involved is obtained by solving formula (3) for P. One must be careful, however, in problems involving several sums, to apply the formula thus obtained only when circumstances justify the application. For example, a formula so obtained should rarely be applied to compute the present value of an annuity. A little reflection will make it clear that the assumption that the same rate of interest will be realized on all interest income as on the principal may well be valid. For since the regular payment of the annuity is greater than the interest on the principal for the

same period, except in the case of a perpetuity, the one who provides the annuity needs in practice to withdraw only enough of the principal which supplemented by the interest will just provide the regular payment of the annuity. If the intervals between the payments of the annuity and the intervals of conversion of interest are the same, this procedure is equivalent to withdrawing the amount of the full payment of the annuity from the principal and then investing the interest at the same rate of interest at which the principal is invested. In any case the solution of the general problem of computing the present value of several sums due at various future times requires a careful analysis of the individual problem and can not be obtained in all cases by employing one general formula. The truth of this statement is further exemplified by the fact that it is apparently impossible to obtain such a formula which is concise and simple to apply, particularly in the case of an annuity.

The problem of finding the present value of a perpetuity whose payments are made at intervals of k years, where the interest payments are to be invested at a different rate of interest from that realized upon the principal, reduces simply to that of finding the principal whose interest payments will accumulate at the second rate to the successive payments of the perpetuity as they become due. If we denote such a principal by A and the successive payments by R, we may write

$$Ais_{\overline{k}|} = R,$$

$$A = \frac{R}{is_{\overline{k}|}}, \tag{7}$$

at rate r, which is the same formula as that ordinarily employed but is to be valued at another rate r (instead of i).

If the principal is invested at rate j payable m times a year and these interest payments and all further interest payments are invested at rate r payable t times a year the principal necessary to provide a perpetuity of R at intervals of k years is

$$A = \frac{R}{is_{\overline{k}|}^{(m)}}. \tag{8}$$

The formula corresponding to three rates of interest would be obtained by replacing $s_{\overline{k}|}^{(m)}$ by that portion of formula (3) included in the brackets minus unity. The formula would be valued, of course, at the third rate of interest.

7. *Valuation of Bonds. Two Rates of Interest.* We shall now derive formulas for valuating a bond which are based upon the assumption that the amortization factors will be invested at another rate of interest than that expected to be realized upon the principal of the bond.

We shall consider first the simplest type of a bond where the face value C is to be repaid in a single sum at the end of n years. If we denote the dividend rate offered in the bond offering by g, the rate to be realized by i, where both interests are payable but once a year, and the corresponding price by A, the amortization factor is $Cg - Ai$. If we treat the amortization factor as the annual payment into a sinking fund which is to accumulate at another rate of interest r and which is to account finally for the reduction in principal or $A - C$, we have

$$(Cg - Ai)s_{\overline{n}|} = A - C,$$

at rate r, where, in the case the bond is bought at a discount, the amortization factor and the reduction in principal are both negative. Solving for A, we find

$$A = C\,\frac{1 + gs_{\overline{n}|}}{1 + is_{\overline{n}|}},$$

at rate r. If we let $C = 1$, A becomes the price per dollar or $1 + k$, where k is the premium or discount per dollar. Solving for k, we obtain

$$k = \frac{g - i}{\dfrac{1}{s_{\overline{n}|}} + i}, \tag{9}$$

at rate r. If it should prove to be possible to invest the amortization factors at the rate planned to be realized on the whole bond, that is, if $r = i$, formula (9) reduces to the formula

$$k = a_{\overline{n}|}(g - i), \tag{B}$$

at rate i, since

$$\frac{1}{s_{\overline{n}|}} + i = \frac{1}{a_{\overline{n}|}},$$

at rate i. Formula (B) is the formula ordinarily employed to value a bond. If all the interests are payable m times a year, the formula for k is easily found to be

$$k = \frac{g - j}{\dfrac{m}{s_{\overline{mn}|}} + j}, \tag{10}$$

at rate r/m, where j is the nominal rate to be realized.

If the principal of a bond is to be repaid in installments it is easily shown that the premium or discount per dollar on the whole bond is simply the weighted arithmetic average of the premiums or discounts per dollar corresponding to the various installments treated as principals of distinct bonds where each premium or discount is weighted by the amount of the corresponding installment. For, if $C_1, C_2, \cdots, C_s$ are the values of the installments and $k_1, k_2, \cdots, k_s$ are the corresponding premiums or discounts per dollar found by one of the formulas derived above, the total price of the whole bond, say $C(1+k)$, where k is the premium or discount per dollar on the whole bond, is given by the formula

$$C(1+k) = C_1(1+k_1) + C_2(1+k_2) + \cdots + C_s(1+k_s).$$

But $C = C_1 + C_2 + \cdots + C_s$. Therefore

$$k = \frac{C_1k_1 + C_2k_2 + \cdots + C_sk_s}{C}. \tag{11}$$

Formulas (10) and (11) can then be employed to value any bond whether the principal is to be repaid in a single sum or in installments.

Formula (10) is especially valuable because it can be applied so easily to solve the inverse problem, namely, given the price of a bond to find the rate of interest which will be realized, since on clearing of fractions j occurs only to the first degree. Those who are familiar with the difficulties encountered in employing other formulas for this purpose will appreciate the advantage to be gained in employing formula (10). This application of formula (10) is so important that the solution of the formula for j is given as follows:

$$j = \frac{g - \dfrac{km}{s_{\overline{mn}|}}}{1+k}, \tag{12}$$

at rate r/m.

Incidentally, if the amortization factors are not invested at all, that is if $r = 0$, formula (12) reduces to the form

$$j = \frac{g - \dfrac{k}{n}}{1+k}. \tag{13}$$

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