THE ISOMORPHISMS OF COMPLEX ALGEBRA.

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Definitions and Conventions. By the *transform* by a function $\phi(x)$ of a function $F(x_1, x_2, \cdots, x_n)$ we mean the expression

$$\varphi\{F(\varphi^{-1}(x_1), \varphi^{-1}(x_2), \cdots, \varphi^{-1}(x_n))\}.$$

We shall consider ∞ as a possible argument for a function, and we shall define $F(\infty)$ to be $\lim\limits_{x\doteq\infty} F(x)$. Similarly if $\lim\limits_{x\doteq\infty} F(x) = \infty$, as $|x|$ grows without limit in every possible way, we shall say that $F(\infty)$ is ∞, while if $\lim\limits_{x\doteq a} F(x) = \infty$ as x approaches a in every possible way, we shall say that $F(a) = \infty$. Similar conventions will be established for functions of more than one variable.

A function F of the complex variable z is said to be *continuous* at z_1 if $\lim\limits_{|\epsilon|=0} F(z_1 + \epsilon) = F(z_1)$.

A function $F(x_1, x_2, \cdots, x_n)$ will be said to be *continuous in general* if there are only a finite number of sets $(x_1, x_2, \cdots, x_n)$ for which $F(x_1, x_2, \cdots, x_n)$ fails to be continuous.

A variable μ is said to *depend uniquely* on $x_1, x_2, \cdots, x_n$ if there are only a finite number of sets $x_1, x_2, \cdots, x_n$ for which μ is not uniquely determined, and if for these μ is undefined.

THEOREM. *If a function Φ is single-valued, as well as its inverse, over the set of arguments consisting of all complex numbers and ∞, and if it is continuous in general, and transforms every algebraic function into an algebraic function, it is a linear function or its conjugate.* The proof of this will involve almost no considerations except those of elementary algebra.

To begin with, let us consider any function of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

which does not degenerate into a constant nor into a function of a single variable. Such a function has the following properties.

(1) $F(x, y)$ depends uniquely on x and y.

(2) x depends uniquely on y and $F(x, y)$.

(3) y depends uniquely on x and $F(x, y)$.

It is easy to show that these properties will also belong to any transform of F by a biunivocal function

Now, no algebraic function not of the form

$$\frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

has properties (1), (2), and (3). Any algebraic function with these properties must be obtained by solving for z an equation $P(x, y, z) = 0$, where P is some polynomial. Since z is uniquely determined by x and y, we may assume, without any real loss of generality, that P is of the form*

$$[g(x, y) + zh(x, y)]^m,$$

which we may write

$$g^m + mg^{m-1}hz + \text{terms in higher powers of } z.$$

Since P is a polynomial, g^m and $g^{m-1}h$ are polynomials. Let us call these Q and R, respectively. P is then of the form

$$\frac{1}{g^{m^2-m}}\{g^m + zhg^{m-1}\}^m = \psi(x, y)\{Q(x, y) + zR(x, y)\}^m,$$

where Q and R may be taken so as to be mutually prime, by removing any common factor and transferring it to the factor ψ. By considerations of symmetry, we may write

$$\begin{aligned} P(x, y, z) &= \psi'(y, z)\{Q'(y, z) + xR'(y, z)\}^{m'} \\ &= \psi''(x, z)\{Q''(x, z) + yR''(x, z)\}^{m''}. \end{aligned}$$

It follows from a consideration of the irreducible factors of P that we may write

$$P(x, y, z) = (\alpha + \beta x + \gamma y + \delta xy - \epsilon z - \xi xz - \eta yz - \vartheta xyz)^m,$$

where $\alpha, \cdots, \vartheta$ are constants. Hence the only algebraic functions satisfying conditions (1), (2), and (3) are of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}.$$

We shall now state a lemma which may readily be established by the usual method of undetermined coefficients. The function $1 - x/y$ is the only function of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

which satisfies the four conditions

* Notice that this is only true in complex algebra.

(1) $F(x, x) = 0$ for all x other than 0 or ∞;
(2) $F(0, x) = 1$ for all x other than 0 or ∞;
(3) $F(x, 0) = \infty$ for all x other than 0 or ∞;
(4) $F\{1, F\{1, F(1, x)\}\} = x$ for all x other than 0 or ∞.

Now, consider a function $G(x, y)$ which is derived from $1 - x/y$ by a transformation Φ which is continuous in general, which is one-to-one, and which leaves all algebraic functions algebraic. We have already shown that any such function $G(x, y)$ must be of the form

$$\frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}.$$

Let us now subject this function to a linear transformation φ which turns the transforms by Φ of 0, 1, and ∞ back into these respective numbers. The resulting function, which we shall call $H(x, y)$, satisfies conditions (1), (2), (3), and (4). Hence we have $H(x, y) = 1 - x/y$. Hence the transformation χ formed by performing first Φ and then φ leaves invariant the function $1 - x/y$. I have shown in an earlier paper* that addition and multiplication can be obtained by the iteration of the function $1 - x/y$. Hence the transformation χ leaves these functions invariant.

Now, any continuous transformation of the complex plane which leaves multiplication invariant must leave invariant the circle of the roots of unity. In a like manner, any continuous transformation of the number-plane that keeps addition invariant must keep invariant first the set of all sets each consisting of all the rational multiples of some number, then the set of all lines each consisting of the real multiples of some number, and finally must turn into a line every line in the complex plane, since every such line can be formed by adding the same number to the product of a given complex number by a variable real number. Hence our transformation χ is an affine transformation which keeps invariant the points 0, 1, and the unit circle. There is no difficulty in showing that any such transformation is either the identity or the conjugate transformation.

It follows that Φ is obtainable by applying after the identity or the conjugate operation the linear operation φ^{-1}. This is precisely the theorem which we set out to prove.

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* *Bilinear operations generating all operations rational in a domain* Ω, ANNALS OF MATHEMATICS, March, 1920.