THE EASTER MEETING OF THE SOCIETY AT CHICAGO.

THE forty-seventh regular meeting of the Chicago Section, constituting the sixteenth regular Western meeting of the Society, was held at the University of Chicago on Friday and Saturday, March 25 and 26, 1921, the first session opening at 10 A.M. on Friday in Room 32, Ryerson Physical Laboratory.

Over sixty persons were present at this meeting, among whom were the following fifty-three members of the Society: Miss Mary C. Ball, Professor G. A. Bliss, Professor Henry Blumberg, Professor R. L. Borger, Dr. H. R. Brahana, Professor C. C. Camp, Professor R. D. Carmichael, Professor E. W. Chittenden, Dr. H. B. Curtis, Professor W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor John Eiesland, Professor Arnold Emch, Professor W. B. Ford, Dr. Gladys Gibbens, Dr. Josephine B. Glasgow, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor T. F. Holgate, Professor Dunham Jackson, Miss Claribel Kendall, Professor A. M. Kenyon, Professor W. C. Krathwohl, Professor Kurt Laves, Mrs. Mayme I. Logsdon, Professor A. C. Lunn, Professor W. D. MacMillan, Professor T. E. Mason, Professor E. D. Meacham, Professor W. L. Miser, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. C. A. Nelson, Professor C. I. Palmer, Professor Anna H. Palmié, Professor H. L. Rietz, Mr. Irwin Roman, Professor J. B. Shaw, Professor H. E. Slaught, Dr. L. L. Steimley, Mr. E. L. Thompson, Professor E. J. Townsend, Professor E. B. Van Vleck, Professor G. E. Wahlin, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young.

Forty-seven persons attended the dinner held at the Quadrangle Club on Friday evening.

At the session of Friday afternoon, presided over by President Bliss, Professor Dunham Jackson read an expository paper on *The general theory of approximation by polynomials and trigonometric sums*. This paper appears in the present number of this BULLETIN. At the close of the discussion, which followed the presentation of this paper, Professor

Wilczynski moved and the Section adopted unanimously a resolution expressing to Professor Jackson their appreciation of the admirable way in which he had presented his subject. In view of the great value of the expository work done by Professor de la Vallée Poussin in the field discussed by Professor Jackson, an address expressing to Professor de la Vallée Poussin their appreciation of his work was signed by those present at the dinner on Friday evening.

The other sessions were presided over by Professor Carmichael, Chairman of the Section, relieved for part of Saturday forenoon by Vice-President Jackson.

The papers presented at the meeting are stated below. Professor Schwatt's papers were communicated to the Society by Professor Dunham Jackson. The papers of Professors Miller and Lane were read by title.

1. Professor I. J. Schwatt: *On the expansion of powers of trigonometric functions.*

No satisfactory expressions for the expansions of powers of trigonometric functions seem to have been given. Only two articles on the subject have been found. One, by Dr. Ely in the AMERICAN JOURNAL OF MATHEMATICS, vol. 5 (1882), p. 359, treats only of the expansion of $\sec^p x$ for odd powers of p and obtains by induction a result which involves Euler's numbers. The other, by Shovelton, in the QUARTERLY JOURNAL OF MATHEMATICS, vol. 46 (1915), pp. 220–247, uses the theory of finite differences.

The author has obtained expansions which are believed to be simple and new.

2. Professor I. J. Schwatt: *On the summation of a trigonometric power series.*

To obtain the sum

$$S \equiv \sum_{n=0}^{\infty} n^q \frac{\sin^p (a + ng)}{(b + nh)!} r^n,$$

the author applies the operator $(r\, d/dr)^q$ to S_1, where $(r\, d/dr)^q$ stands for the repetition q times of the operator rd/dr, r and d/dr not being permutable, and where S_1 is of the same form as S except that n^q is wanting. Next S_1 is reduced to

$$S_2 = \sum_{n=0}^{\infty} \frac{\rho^n}{(b + nh)!},$$

where $\rho = re^{(p-2k)gi}$ $(k = 0, 1, 2, \cdots, p)$; and an expression for S_2 is obtained by introducing a function $f(n)$ defined by the formula

$$f(n) = \frac{x^n}{(b+n)!}\sum_{k=1}^{h}\theta_k{}^n,$$

where θ_k is one of the kth roots of unity.

3. Professor W. B. Ford: *A disputed point regarding the nature of the continuum.*

In this paper the author considers the question, originally proposed by Du Bois-Reymond, as to whether a decimal whose digits are chosen in an arbitrary manner (as by the throwing of a die) can properly be regarded as a real number. The question forms but a special example of various questions of broader significance which were extensively discussed by Borel, Hadamard, König and others during the decade preceding the war, their conclusions, however, being widely at variance. It is merely desired in the present paper to point out that such numbers must be regarded as acceptable in case one is to regard the familiar cardinal number theory of Cantor, particularly as regards the continuum, as having a proper place in mathematics, since the derivation of Cantor's results clearly depends upon the use of decimals whose digits are assigned entirely at random.

4. Mrs. Mayme I. Logsdon: *The equivalence of pairs of hermitian forms.*

In this paper the author makes a generalization of the theory of elementary divisors for hermitian λ-matrices with special application to pairs of hermitian forms. If two hermitian matrices A and B are equivalent in the sense that there exist non-singular matrices C and D with determinants free of λ such that $B = CAD$, it is shown that there exists a non-singular matrix P with determinant free of λ such that $B = \bar{P}'AP$, where $\bar{P}$ means the matrix formed from P by replacing each element by its conjugate imaginary and P' means P transposed. It is found that in the case of hermitian forms as well as bilinear and quadratic forms the coincidence of the elementary divisors is a necessary and sufficient condition for equivalence.

The Weierstrass reduction is found to hold where one of the forms is definite, thereby insuring reality of all the elementary divisors. When complex elementary divisors are present they enter in conjugate pairs. In this case it was found possible to regularize the matrix with respect to the two conjugate linear factors simultaneously and to expand simultaneously with respect to these two factors the terms representing the adjoint form. A simple canonical form was obtained.

5. Mr. C. C. MacDuffee (introduced by Professor L. E. Dickson): *Invariants and vector covariants of linear algebras without the associative law.*

A vector covariant of a linear algebra has been defined by Professor Hazlett (TRANSACTIONS, vol. 19) as a covariant which involves one or more of the units $e_1, \cdots, e_n$. In this paper the author considers the general linear algebra in which the commutative and associative laws of multiplication are not assumed, and shows that, after a vector covariant of weight μ has been expressed linearly in the units, the coefficients of the units are transformed cogrediently, apart from the factor $D^{-\mu}$, with the coefficients of the general number of the algebra under a linear transformation of determinant D. Therefore the coefficients of a vector covariant may be inserted in place of the variables in the covariants of the characteristic equations to give additional relative invariants. After defining a determinant whose elements are hypercomplex numbers for which multiplication is not assumed to be commutative or associative, and after showing that certain elementary theorems concerning determinants apply to these hypercomplex determinants, it is proved that the hypercomplex determinant each of whose n rows is $e_1, e_2, \cdots, e_n$ is a vector covariant of weight 1 for every manner of grouping the factors in each term.

6. Professor E. J. Wilczynski: *Some projective generalizations of geodesics.*

The two most important properties of geodesics are: (*a*) their connection, as extremal curves, with the length of arc integral; (*b*) the property that the osculating plane at any point on such a curve contains the corresponding surface normal.

In this paper, the author considers the generalizations which arise from these two points of view; (*a*) by replacing the length of arc integral by some other integral invariant of the same general form; (*b*) by replacing the congruence of normals by some other congruence. The two classes of curves obtained in this way do not coincide; but both are included in a larger class of two-parameter curves defined by a second order differential equation of the form

$$\frac{d^2v}{du^2} = M\left(\frac{dv}{du}\right)^3 + N\left(\frac{dv}{du}\right)^2 + P\frac{dv}{du} + Q,$$

where M, N, P, and Q are functions of u and v, the parameters of a surface point. These classes of curves may be characterized by properties of the osculating planes of the curves of a family which pass through a given surface point and dualistically related properties. It then remains to locate the two more special classes mentioned above within the larger class.

7. Professor W. L. Hart: *Summable infinite determinants.*

It has been customary to define the value of an infinite determinant (1) $D = |a_{ij}|$ $(i, j = 1, 2, \cdots)$ by the equation $D = \lim_{n=\infty} D_n$, in case the limit exists, where D_n is the determinant of order n formed by the elements of D for $i, j = 1, 2, \cdots, n$. Utilizing Cesàro summability of order r, the author assigns the value D to the determinant (1) if the sequence $(D_n; n = 1, 2, \cdots)$ is summable to the value D. The results of the paper are restricted to the case $r = 1$. If in (1) we choose all elements as zero except for $a_{2k-1,\, 2k} = 1$ and $a_{2k,\, 2k-1} = -1$ $(k = 1, 2, \cdots)$, D is summable to the value $\frac{1}{2}$. If quantities c_{ij} are added to the elements of this determinant, then, in case $\Sigma|c_{ij}|$ converges, the new determinant thus obtained is summable and possesses most of the useful properties of normal infinite determinants. There is also exhibited a more complicated type of summable determinants, related in a simple way to a determinant (1), summable to the rational value p/q $(p < q)$, all of whose elements are either $+1$, -1, or 0. It should be noted that no cases in the solution of infinite systems of linear equations can be solved by means of summable determinants which are not also solvable by means of normal infinite determinants.

8. Professor Henry Blumberg: *New properties of all functions.*

For the sake of simplicity the following statements will be made to refer to functions $z = f(x, y)$ of two variables. A point (ξ, η, ζ), where $\zeta = f(\xi, \eta)$, of the surface $z = f(x, y)$

is said to be *densely approached* if for every $\gamma > 0$ two numbers $\alpha > 0$ and $\beta > 0$ exist such that the projections upon the XY-plane of the surface points in the rectangular parallelopiped $\xi - \alpha \leqq x \leqq \xi + \alpha$, $\eta - \beta \leqq y \leqq \eta + \beta$, $\zeta - \gamma \leqq z \leqq \zeta + \gamma$ are everywhere dense in the rectangle $\xi - \alpha \leqq x \leqq \xi + \alpha$, $\eta - \beta \leqq y \leqq \eta + \beta$. The author has found that the points (x, y) for which (x, y, z) is densely approached constitute a residual set (set of second category, according to Baire) of the XY-plane; conversely, for every residual set R, there exists a function $f(x, y)$ such that (x, y, z) is densely approached if and only if (x, y) is in R. It follows that for every function $f(x, y)$ a dense set D of the XY-plane exists, such that f is continuous if it is defined only in D. The results flow from a general theorem concerning *closed region-point* relations. There are generalizations for n dimensions, function space and more general spaces.

9. Professor E. B. Van Vleck: *On non-loxodromic substitution groups in n dimensions.*

This paper is a continuation of a previous paper. For linear substitutions $z' = (az + b)/(cz + d)$ without a common pole there exist two types of non-loxodromic groups, the one consisting of non-loxodromic substitutions having a common invariant circle, while the other contains exclusively elliptic substitutions which are without a common invariant circle. On the author's definition of a non-loxodromic substitution in n dimensions corresponding results hold, but the analog of the elliptic group exists only in space of an odd number of dimensions.

10. Professor G. A. Miller: *An overlooked infinite system of groups of order pq^2, p and q being prime numbers.*

Lists of the possible abstract groups of order pq^2, p and q being prime numbers, were published almost simultaneously by Cole and Glover in the AMERICAN JOURNAL OF MATHEMATICS, vol. 15 (1893), and by O. Hölder in the MATHEMATISCHE ANNALEN, vol. 43 (1893). In a subsequent article published in volume 46 of the latter journal, Hölder directed attention to the fact that the enumeration of these groups contained in the first article mentioned above was incomplete. In the present article the author directs attention to the fact that Hölder's enumeration is also incomplete and observes that the same incompleteness appears in the enumerations found in the first and also in the second edition of W. Burnside's *Theory of Groups of Finite Order* (1897 and 1911), as well as elsewhere.

There is a system which contains $q - 1$ distinct abstract groups for a given value of p and q, while only one of these groups is found in the published lists.

11. Professor L. E. Dickson: *Fallacies and misconceptions in diophantine analysis.*

This paper has appeared in the April number of this BULLETIN.

12. Professor L. E. Dickson: *A new method in diophantine analysis.*

This paper has appeared in the May number of this BULLETIN.

13. Professor T. H. Hildebrandt: *On a general theory of functions.* (Preliminary report.)

In recent years there have been a number of attempts in the direction of extending the results of the theory of functions of n variables to other cases, that of infinitely many variables, and functionals of continuous functions having received the most attention. This work has been done chiefly by Fréchet, Hart and Gateaux. In this paper an attempt is made to extend by generalization the domain of application of the results already attained, and derive new ones in the general domain. The author uses the basis recently suggested by Lamson (AMERICAN JOURNAL OF MATHEMATICS, vol. 42 (1920), p. 245). So far there have been obtained theorems of mean value, a theorem on implicit functions, existence theorems of differential equations, and some theorems on analytic functions.

14. Professor Arnold Dresden: *Some new formulas in combinatory analysis.*

In this paper the author presents some formulas involving binomial coefficients, which proved of importance in work on symmetric functions with which he is occupied. A search of the literature of combinatory analysis has failed to bring these formulas to light. They are therefore believed to be new.

15. Professor J. B. Shaw: *Generational definition of linear associative hypernumbers.*

Let the n^2 hypernumbers λ_{ij} $(i, j = 1, \cdots, n)$ and certain nilpotent hypernumbers θ_{rs} be given; then the general linear associative hypernumber is of the form

$$\Sigma\lambda_{ij}A_{ij}(\theta_{aj})B_{ijk}(\theta_{bk})C_{ijkl}(\theta_{cl}) \cdots.$$

The laws by which the λ's and the θ's may be permuted in a product are discussed in this paper. They are essential to the algebra to which the general hypernumber belongs.

16. Professor J. B. Shaw: *On Hamiltonian products. Second paper.**

Our range is the totality of functions of a single real variable such that the product of any two is integrable (Lebesgue, Stieltjes, etc.) over a given interval. Each function is represented with a different argument save as specified. Let $\alpha_1(s_1), \alpha_2(s_2), \cdots$ be functions of the totality considered. Then there are two processes considered to be fundamental. The first consists in constructing out of n functions a function of n arguments as follows. Write the product of the n functions $\alpha_1(s_1)\alpha_2(s_2)\cdots\alpha_n(s_n)$. Form similar products from this by permuting the arguments only into every possible arrangement, giving the result a sign $+$ or $-$ according to the inverions, as in determinants. The results are then added. This is represented by $A_n\alpha_1\alpha_2\cdots\alpha_n$.

The second fundamental process consists in starting from the product $\alpha_1(s_1)\alpha_2(s_2)\cdots\alpha_n(s_n)$ as before, but now forming products by making two arguments identical in every possible manner, the sign of the term being determined by the number of inversions necessary to bring the two functions containing the identical arguments together. Each term is then integrated as to the equal arguments over the given interval. If we add the results, we obtain a function of $n - 2$ arguments indicated by $A_{n-2}\alpha_1\alpha_2\cdots\alpha_n$. Repeating the process, or what is the same thing, making two pairs of arguments identical in every possible way, and then integrating as to each of the two, determining signs as before, leads to the function $A_{n-4}\alpha_1\alpha_2\cdots\alpha_n$. Then we define the Hamiltonian product of the functions $\alpha_1, \alpha_2, \cdots, \alpha_n$ as the function

* See this BULLETIN, (2), vol. 16 (1910), p. 304.

$H[\alpha_1(s_1),\ \alpha_2(s_2),\cdots,\ \alpha_n(s_n)] = A_n\alpha_1\alpha_2\cdots\alpha_n + A_{n-2}\alpha_1\alpha_2\cdots\alpha_n + \cdots + A_1\alpha_1\alpha_2\cdots\alpha_n$ (or $A_0\alpha_1\alpha_2\cdots\alpha_n$) for the last term. The author considers the resulting theorems.

17. Professor F. E. Wood: *Congruences characterized by certain coincidences.*

The equations of certain lines covariantly related to a general congruence of lines have previously been obtained with respect to various tetrahedra of reference. In this paper the author obtains the equations of these lines with respect to a single fundamental tetrahedron of the congruence as well as the general equations which transform the equations of any configuration related to either of two other tetrahedra of reference into the equations of the same configuration related to this fundamental tetrahedron.

The possible coincidences of these lines, for every line of the congruences, are studied in detail. The congruences characterized by certain coincidences are obtained in a canonical form. In particular, the class of congruences for which the directrix of the first kind of each focal sheet coincides with the directrix of the second kind of the other sheet is found to be the same as the class of congruences associated with functions of a complex variable by Wilczynski.

18. Professor E. P. Lane: *A general theory of congruences.*

Geometers who have considered conjugate nets from the point of view of projective differential geometry have for the most part been content to study a single net, together with its attendant configurations, and have been forced to carry out a tedious transformation whenever it has been desired to consider another conjugate net on the same surface.

The author lays the foundation for a theory of all the conjugate nets on a surface, using the following method. The asymptotic net is taken as parametric. Then the fundamental invariants and covariants of an arbitrary conjugate net are calculated in terms of the asymptotic parameters. The resulting formulas are comparatively simple, and their usefulness is demonstrated by applications to pencils of nets, harmonic nets, isothermally conjugate nets, and plane nets.

Among other theorems, it is proved that if there are more than three harmonic nets in one pencil, the pencil is an isothermally conjugate quadratic pencil. And if there are exactly three harmonic nets in a pencil, the pencil is the Segre-Darboux pencil. It is shown that the envelope of the ray of a point for all nets of a pencil is a conic.

19. Professor John Eiesland: *The group of motions of an Einstein space.*

The question to what extent an Einstein space is determined by its group of motions seems to be of interest both from a geometrical and from a physical standpoint. Three assumptions L, M, and N have been made. L is of group-theoretical nature and postulates that the general space $\Sigma_1^4 g_{ik} dx_i dx_k$ admits a certain 4-parameter intransitive group of motions which keeps the origin of the space $t = const.$ fixed. The general space admitting the group G_4 is shown to be of the form

$$(1) \quad -ds^2 = (1 + \phi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) - 2\varphi_3 dRdt - \varphi_1 dt^2,$$

where φ_1, φ_2, φ_3 are functions of R. The static space, for which $\varphi_3 = 0$, or,

$$(2) \quad -ds^2 = (1 + \varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi^1 dt^2,$$

is then considered. Adopting two additional assumptions, M and N, we arrive at Schwarzschild's form. The two assumptions are as follows. M. The discriminant of (2) is invariant and equal to $R^4 \sin^2(\theta - c^2)$. (This assumption is also known as Kottler's fundamental hypothesis.*) N. The sum of the principal Riemannian curvatures of the space

$$ds_0^2 = (1 + \varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

is equal to zero.

By varying the last assumption other forms of approximately Euclidean spaces have been obtained. Thus, if the sum is inversely proportional to the fourth power of R, we obtain Weyl's line-element for the case of a spherical mass with a constant electrical charge. In the final part of the paper is considered a gravitational space admitting a two-parameter group of motions, namely a translation along the t-axis and a rotation about the z-axis.

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Secretary of the Chicago Section.

* F. Kottler, *Grundlagen der Einsteinschen Gravitationstheorie*, ANNALEN DER PHYSIK, (4), vol. 56, p. 409.