# PARAMETRIC EQUATIONS OF THE PATH OF A PROJECTILE WHEN THE AIR RESISTANCE VARIES AS THE $n$TH POWER OF THE VELOCITY.

BY PROFESSOR F. H. SAFFORD.

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THE differential equations to be solved are

$$(1) \qquad \frac{W}{g}\frac{d^2x}{dt^2} = -Kv_c^n\frac{dx}{ds}, \qquad \frac{W}{g}\frac{d^2y}{dt^2} = W - Kv_c^n\frac{dy}{ds},$$

in which $v_c$ is the velocity along the path, $W$ is the weight of the projectile and $K$ and $n$ are experimental constants, the dimensions of $K$ being $W \cdot l^{-n} \cdot t^n$. Obviously the $X$ axis is taken horizontal, the $Y$ axis vertically downward.

M. de Sparre* gives a solution for $n = 2$, making however certain approximations in the early stages, and presents his results in two cases corresponding to the paths before and after the time when the slope is unity. Greenhill† treats with much detail the case of $n = 3$.

For the general case of unrestricted $n$, equations (1) may be written

$$(2) \qquad \begin{aligned} \frac{W}{g}\frac{d^2x}{dt^2} &= -K\left[\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right]^{(n-1)/2} \cdot \frac{dx}{dt}, \\ \frac{W}{g}\frac{d^2y}{dt^2} &= W - K\left[\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right]^{(n-1)/2} \cdot \frac{dy}{dt}, \end{aligned}$$

and next transformed by writing

$$(3) \qquad \frac{dx}{dt} = v = r\cos\theta, \qquad \frac{dy}{dt} = u = r\sin\theta,$$

so that $r$ and $\theta$ are the polar coordinates of the hodograph. If the origin is taken at the point of release of the projectile and $\alpha$ is the angle of depression, $V$ being the initial velocity,

* *Comptes Rendus*, volume 160, p. 584.

† Elliptic Functions, pp. 244–53.

then for $x$, $y$ and $t$ all zero, $r$ and $\theta$ become $V$ and $\alpha$ respectively, also

$$(4) \qquad \frac{dx}{dt} = V \cos \alpha, \qquad \frac{dy}{dt} = V \sin \alpha.$$

Writing $W/(gK) = S$, $W/K = T$, equations (2) become successively

$$(5) \quad S\frac{dv}{dt} = -(u^2+v^2)^{(n-1)/2}\cdot v, \qquad S\frac{du}{dt} = T-(u^2+v^2)^{(n-1)/2}\cdot u,$$

$$(6) \qquad S\frac{dr}{dt} = T\sin\theta - r^n, \qquad Sr\frac{d\theta}{dt} = T\cos\theta,$$

$$(7) \qquad \frac{dr}{T\sin\theta - r^n} = \frac{rd\theta}{T\cos\theta} = \frac{dt}{S}.$$

From (7)

$$\frac{dr}{d\theta} - r\tan\theta = -\frac{r^{n+1}}{T\cos\theta}$$

which is reducible to the linear form, giving

$$(8) \qquad r = \left[\frac{nH}{T} + \frac{n}{T}\int_0^\theta \sec^{n+1}\theta d\theta\right]^{-(1/n)} \sec\theta.$$

$H$ in (8) is a constant of integration to be determined by the use of the initial conditions, i.e., $r$ becomes $V$ when $\theta$ is $\alpha$, hence

$$(9) \qquad V = \left[\frac{nH}{T} + \frac{n}{T}\int_0^\alpha \sec^{n+1}\theta d\theta\right]^{-(1/n)} \cdot \sec\alpha$$

or

$$(10) \qquad H = \frac{W}{nKV^n\cos^n\alpha} - \int_0^\alpha \sec^{n+1}\theta d\theta.$$

(It will be noticed that $H$ is of zero dimension.) Later expressions will be simplified by the introduction of a new constant $N$, which becomes unity when $\alpha$ is zero, or the initial direction of the motion is horizontal, viz.,

$$N = \left[1 + \frac{1}{H}\int_0^\alpha \sec^{n+1}\theta d\theta\right]^{1/n} \cos\alpha,$$

but $N$ need not be computed since

$$(11) \qquad VN = (W/(nKH))^{1/n}.$$

The reduced form of (8) is

$$(12)\qquad r = VN\left[1 + \frac{1}{H}\int_0^\theta \sec^{n+1}\theta d\theta\right]^{-(1/n)} \sec\theta.$$

Again from (7)

$$(13)\quad dt = \frac{Srd\theta}{T\cos\theta} = \frac{VN}{g}\left[1 + \frac{1}{H}\int_0^\theta \sec^{n+1}\theta d\theta\right]^{-(1/n)} \sec^2\theta d\theta.$$

Combining (3) and (13),

$$(14)\qquad \begin{aligned} dx &= \frac{V^2N^2}{g}\left[1 + \frac{1}{H}\int_0^\theta \sec^{n+1}\theta d\theta\right]^{-(2/n)} \sec^2\theta d\theta, \\ dy &= \frac{V^2N^2}{g}\left[1 + \frac{1}{H}\int_0^\theta \sec^{n+1}\theta d\theta\right]^{-(2/n)} \sec^2\theta \tan\theta d\theta. \end{aligned}$$

In (14) let $\tan\theta = z$, thus introducing $z$ as the parameter which will appear in the final equations, whence

$$(15)\qquad \begin{aligned} dx &= \frac{V^2N^2}{g}\left[1 + \frac{1}{H}\int_0^z (1 + z^2)^{(n-1)/2}dz\right]^{-(2/n)} dz, \\ dy &= \frac{V^2N^2}{g}\left[1 + \frac{1}{H}\int_0^z (1 + z^2)^{(n-1)/2}dz\right]^{-(2/n)} zdz. \end{aligned}$$

Integration of (15) completes the formal task of obtaining the parametric equations of the trajectory, equations (10) and (11) giving the values of $H$ and $VN$ respectively. The practical difficulty lies in obtaining expansions of the bracketed expression in (15).

As a preliminary it is necessary to divide the computation (and the path) into two parts corresponding to values of $z$ less than and greater than a certain value $z_0$, defined by

$$(16)\qquad \frac{1}{H}\int_0^{z_0} (1 + z^2)^{(n-1)/2}dz = 1,$$

a procedure which was suggested by a discussion in Chrystal's Algebra, volume 2, page 213. The computation of $z_0$ is by approximation, and if $z_0$ is less than 1 the binomial theorem is available. When $z_0$ is greater than 1 the integration in (16) must be taken in two intervals and that from 1 to $z_0$ effected by replacing $z$ by $s^{-1}$ before expanding, also using $s_0 = z_0^{-1}$.

Let $P$ be the bracketed expression in (15), and then $\varphi(P) = [P(z)]^{-(2/n)}$ and $\varphi(Q) = [Q(s)]^{-(2/n)}$ can represent functions in (15) in the respective intervals 0 to $z_0$ for $z$ and 0 to $s_0$ for $s$. Arbogast's method of derivations (v. Williamson's Differential Calculus) is of assistance in making the necessary expansions especially as the computations for the two functions are identical up to a certain stage. Following Arbogast's notation closely, the work for $\varphi(P)$ is as follows:

$$(17)\quad \varphi(P) = \left[1 + \frac{1}{H}\int_0^z (1+z^2)^{(n-1)/2}dz\right]^{-(2/n)} = A + Bz + Cz^2/2! + Dz^3/3! + \cdots,$$

$$(18)\quad \begin{aligned} P(z) &= a + bz + cz^2/2! + dz^3\,3! + \cdots \\ &= P(0) + P'(0)z + P''(0)z^2/2! + P'''(0)z^3/3! + \cdots, \end{aligned}$$

in which $z$ lies between 0 and $z_0$, and $A, B, \cdots a, b$ are to be computed. From (17) and (18) come the successive derivatives of $\varphi(P)$, also

$$(19)\quad \begin{aligned} a = P(0) = 1, \quad b = P'(0) = 1/H, \quad c = P''(0) = 0, \\ d = P'''(0) = (n-1)/H \cdots, \end{aligned}$$

$$(20)\quad \begin{aligned} A = \varphi(a), \quad B = \varphi'(a)\cdot b, \quad C = \varphi'(a)\cdot c + \varphi''(a)\cdot b^2, \\ D = \varphi'(a)\cdot d + \varphi''(a)\cdot 3bc + \varphi'''(a)\cdot b^3 \cdots, \end{aligned}$$

$$(21)\quad \begin{aligned} \varphi(a) = 1, \quad \varphi'(a) = -2/n, \quad \varphi''(a) = 2(2+n)/n^2, \\ \varphi'''(a) = -2(2+n)(2+2n)/n^3 \cdots. \end{aligned}$$

These give the desired coefficients of $\varphi(P)$, viz.,

$$(22)\quad \begin{aligned} A = 1, \quad B = -2/(nH), \quad C = 2(2+n)/(n^2H^2), \\ D = 2(n-1)/(nH) - 2(2+n)(2+2n)/(n^3H^3) \cdots. \end{aligned}$$

Similarly by the aid of (16) come

$$(23)\quad \begin{aligned} Q(s) &= 2 + \frac{1}{H}\int_s^{s_0} \frac{(1+s^2)^{(n-1)/2}}{s^{n+1}}\,ds, \\ Q'(s) &= -\frac{1}{H}\,\frac{(1+s^2)^{(n-1)/2}}{s^{n+1}}, \end{aligned}$$

$$(17)^*\quad \varphi(Q) = \bar{A} + \bar{B}(s-s_0) + \bar{C}(s-s_0)^2/2! + \bar{D}(s-s_0)^3/3! + \cdots,$$

$$(18)^*\quad Q(s) = \bar{a} + \bar{b}(s-s_0) + \bar{c}(s-s_0)^2/2! + \bar{d}(s-s_0)^3/3! + \cdots$$

(in which $s$ lies between 0 and $s_0$). $\varphi(Q)$, $\varphi'(Q)$, $\cdots$ are of the same form as $\varphi(P)$, $\varphi'(P)$, $\cdots$. From (17)* and (18)* come

(19)* $\bar{a} = Q(s_0) = 2,\ \bar{b} = Q'(s_0) = -(1+s_0^2)^{(n-1)/2}/(Hs_0^{n+1}) \cdots.$

$\bar{A}, \bar{B}, \bar{C}, \cdots$ in terms of $\bar{a}, \bar{b}, \bar{c}, \cdots$ follow from (20). From (19)*,

(21)* $\varphi(\bar{a}) = 2^{-(2/n)}, \qquad \varphi'(\bar{a}) = -2(2)^{-(2+n)/n}/n \cdots,$

giving finally $\bar{A}, \bar{B}, \bar{C}, \cdots$ in terms of $n$, $s_0$ and $H$.

After obtaining $A, B, \cdots; \bar{A}, \bar{B}, \cdots$ it is necessary to substitute (17) and (17)*, corresponding respectively to the first and second portions of the path, in (15) and then to integrate. It should be noticed that $P(z_0)$ and $Q(s_0)$ are identical, so that the two portions have the same slope at their common point. For the first portion of the path

$$(24)\quad \begin{aligned} x &= \frac{V^2N^2}{g}[Az + Bz^2/2! + Cz^3/3! + Dz^4/4! \cdots], \\ y &= \frac{V^2N^2}{g}[Az^2/2 + Bz^3/3 + Cz^4/4\cdot 2! + Dz^5/5\cdot 3! \cdots], \end{aligned}$$

no constant of integration being added as the origin is on the path. For the second portion of the path, using $s = 1/z$ as the parameter,

$$(25)\quad \begin{aligned} x &= \frac{V^2N^2}{g}\left[\bar{X} - \bar{A}\int_{s_0}^{s}\frac{ds}{s^2} - \bar{B}\int_{s_0}^{s}\frac{(s-s_0)}{s^3}ds \cdots\right], \\ y &= \frac{V^2N^2}{g}\left[\bar{Y} - \bar{A}\int_{s_0}^{s}\frac{ds}{s^3} - \bar{B}\int_{s_0}^{s}\frac{(s-s_0)}{s^4}ds \cdots\right]. \end{aligned}$$

$\bar{X}$ and $\bar{Y}$ are constants of integration having the respective values of the two bracketed expressions in (24) when $z$ has the value $z_0$, i.e., when $s$ is $s_0$, so that the two portions of the path join.

It is evident that the accuracy of the observed constants $W, K, V, n$ affects that of the computed constants, $H, N, \bar{X}, \bar{Y}$, and fixes a point beyond which the series expansions need not be carried, while the fact that $x$ and $y$ ultimately vary as $V^2$ shows that a given percentage of error in $V$ gives approximately a double percentage of error in $x$ and $y$.

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