THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The two hundred and fifth regular meeting of the Society was held at Columbia University on Saturday, October 25, 1919. The following forty-eight members attended the two sessions:

Dr. J. W. Alexander, Professor R. C. Archibald, Dr. I. A. Barnett, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Pierre Boutroux, Dr. G. A. Campbell, Professor F. N. Cole, Dr. Tobias Dantzig, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. H. Gronwall, Professor C. O. Gunther, Professor Olive C. Hazlett, Dr. A. A. Himwich, Mr. S. A. Joffe, Professor Edward Kasner, Mr. H. P. Kean, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. K. W. Lamson, Professor P. H. Linehan, Professor James Maclay, Professor R. L. Moore, Professor G. W. Mullins, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor D. E. Smith, Professor P. F. Smith, Dr. J. M. Stetson, Mr. J. J. Tanzola, Professor H. D. Thompson, Professor Oswald Veblen, Mr. A. C. Washburne, Mr. H. E. Webb, Professor J. H. M. Wedderburn, Professor Mary E. Wells, Mr. R. A. Wetzel, Mr. J. K. Whittemore, Professor A. H. Wilson.

Vice-President G. D. Birkhoff occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. C. C. Camp, University of Illinois; Professor C. J. Coe, University of Michigan; Dr. Teresa Cohen, Johns Hopkins University; Mr. W. E. Heal, U. S. Bureau of Plant Industry; Dr. C. A. Nelson, University of Kansas; Mr. J. L. Walsh, Harvard University. Four applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. Propositions for establishing a board of custodians of the property of the Society and for issuing a new catalogue of the library were laid over for action at the annual meeting.

The following papers were read at the October meeting:

(1) Mr. J. K. WHITTEMORE: "Motion in a resisting medium."

(2) Miss ANNA M. MULLIKIN: "A countable collection of mutually exclusive closed point sets with connected sum."

(3) Professor OLIVE C. HAZLETT: "New proofs of certain finiteness theorems in the theory of modular covariants."

(4) Professor R. D. CARMICHAEL: "On the convergence of certain classes of series of functions."

(5) Professor R. D. CARMICHAEL: "Note on a transformation of series similar to the principle of inversion in the theory of numbers."

(6) Professor R. D. CARMICHAEL: "Note on the theory of integral functions of the first class."

(7) Professor ANNA J. PELL: "Two linear integral equations with two parameters."

(8) Professor G. D. BIRKHOFF: "Note on stable periodic orbits."

(9) Professor O. D. KELLOGG: "Some problems connected with submarine acoustics."

(10) Dr. T. H. GRONWALL: "Differential variations in ballistics, with applications to the qualitative properties of the trajectory."

(11) Professor A. A. BENNETT: "Standard density, temperature and pressure of air aloft."

(12) Professor A. A. BENNETT: "The probable error of a small number of rounds."

(13) Professor A. A. BENNETT: "The physical bases of ballistic table computations."

(14) Professor A. A. BENNETT: "The sign of the distance in analytical geometry."

(15) Dr. E. F. SIMONDS: "Invariants of infinite groups in the plane."

(16) Professor EDWARD KASNER: "The motion of n bodies, under any forces, starting from rest."

The paper of Miss Mullikin was communicated to the Society and read by Professor R. L. Moore. Dr. Simonds's paper was read by Professor Kasner. The papers of Professor Carmichael and the last three papers of Professor Bennett were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Whittemore's paper is devoted to the study of the motion of a particle whose tangential acceleration is a given function of the velocity alone, $a = F(v)$. It is proved that if λ is a simple positive root of $F(v) = 0$ and $F(v)$ is subject to certain other conditions, generally satisfied in a real problem, $\lambda - v$ approaches zero at least as rapidly as e^{-mt}, where t is the time and m a positive constant; that under the same conditions $\lambda t - x$, where x is the distance moved, approaches a limit L, and differs from L by a quantity not greater than a constant multiple of e^{-mt}. When the root λ is not simple the results are different. Various examples are considered and some numerical illustrations given. Four problems are discussed in which there act forces working at a constant rate. Such problems seem to have been neglected in books on elementary mechanics, but have nevertheless some interest. Especial study is made of the starting of a steamer driven by engines developing constant horsepower.

2. The sum of a finite number (more than one) of mutually exclusive closed point sets is never connected.* Nor is the sum of a countably infinite number of such point sets connected if each of them consists of only one point. In one dimension no countably infinite collection of mutually exclusive closed point sets ever has a connected sum. One might rather naturally be inclined to believe that this proposition holds true also in two dimensions. Miss Mullikin shows by an example that this is, however, not the case.

3. Professor Hazlett's paper gives new proofs of certain finiteness theorems in the theory of modular invariants. In 1909 Professor Dickson proved the finiteness theorem for modular invariants in the Galois field $GF[p^n]$. This was followed in 1913 by his proof of the finiteness theorem for modular covariants. The next year, Professor Wiley proved the finiteness theorem for modular invariants of a system of forms over the Galois field $GF[p]$ and any number of cogredient points (x_i, y_i). This proof used a lemma which is closely related to Hilbert's well known theorem on a set of polynomials.

* A set of points is said to be connected if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

In a paper presented at Chicago in December, 1918, the author proved a theorem which enables us to construct all modular covariants of a system S of forms from the modular invariants of an enlarged system S' consisting of the forms of S and an additional linear form. By using this theorem, and the finiteness theorem for modular invariants, we can prove at once the finiteness theorem for modular covariants as well as the more general theorem proved by Professor Wiley. The present proof gives a method for obtaining from a fundamental set of invariants of S' a set of modular covariants of S in terms of which all modular covariants of S are expressible as polynomials.

Finally, by means of a symbolic notation, the author proves from this theorem that every formal covariant of a system of forms over the Galois field $GF[p^n]$ is a polynomial in the modular covariants which have been made formally invariant as to the coefficients of the forms and in the irreducible covariants which are congruent to zero whenever the coefficients are marks of the field. This theorem is verified for the binary quadratic modulo 3 and for the binary cubic modulo 2.

4. In this paper Professor Carmichael develops the first fundamental convergence properties of certain very general classes of series including many of those which are classic in analysis, such as the power series, factorial series, and Dirichlet series.

5. A certain general principle of transformation of series analogous to the principle of inversion in the theory of numbers is here employed by Professor Carmichael to derive a few classic results; and the principle itself is extended and more general results are obtained through use of generalizations of the λ-function often employed in the theory of the distribution of prime numbers.

6. The object of this note by Professor Carmichael is to extend certain of the classic theorems about integral functions. It contains a modification of the Weierstrass factor theorem valid in certain cases and some consequences relative to the relation between the zeros of an integral function and those of its derivative.

7. It is shown in Mrs. Pell's paper that there exist real values of λ and μ for which the two linear integral equations

$$u(x) = \lambda \int_a^b K(x, y)u(y)dy + \mu \int_a^b L(x, y)u(y)dy,$$

$$v(s) = \lambda \int_c^d M(s, t)v(t)dt - \mu \int_c^d N(s, t)v(t)dt$$

have solutions $u(x) \not\equiv 0$ and $v(s) \not\equiv 0$, if K, L, M, N are continuous and symmetric and L, N are positive definite. Functions of two variables, which can be expressed in a certain form, can be expanded into a uniformly convergent series in terms of the functions of a system adjoint to the system $\{u_i(x)v_i(s)\}$.

8. The object of this note by Professor Birkhoff is to establish that infinitely many periodic orbits of certain types exist in the infinitesimal vicinity of any stable periodic orbit.

The note will appear in the *Proceedings of the National Academy of Sciences.*

9. Professor Kellogg outlined a number of mathematical problems which have arisen in connection with the detection of submerged submarines by sound.

10. Dr. Gronwall's paper gives a method for computing differential variations, based on the discovery of a new algebraic integral of Bliss's adjoint system of differential equations. By means of this integral, the third order system of Bliss reduces to a linear differential equation of the second order, and the numerical computation of the differential variations is materially shortened both for ordinary and anti-aircraft trajectories.

The linear differential equation referred to being of the second order, the general behavior of its solutions can be determined without much difficulty, and there result a large number of qualitative properties of the trajectory, for instance the following: the lateral deflection of the projectile due to a constant cross wind is always less than the increase in range due to a following wind of the same velocity.

11. This paper by Professor Bennett discusses the special question of the choice of certain functions assumed to represent standard density, temperature and pressure of air aloft as functions of the altitude measured in meters. The necessity for such standard functions and the reason for the exponential forms selected as against meteorological means, are explained, as applied to the problem of exterior ballistics.

12. Many known variables are taken explicitly into account to secure as great accuracy as possible in firing heavy artillery. There remains, however, an inevitable dispersion, upon which also computations are based through the magnitude of the probable errors in range and deflection. As an ordnance problem, the probable error of the various types of ammunition must be measured. A consideration of the accuracy of the probable error when based for the sake of economy on a relatively small number of rounds is here investigated by Professor Bennett, along the lines laid out by Czuber in his standard treatises Beobachtungsfehler and Wahrscheinlichkeitsrechnung. The paper will appear shortly in the *Journal of the U. S. Artillery.*

13. A pamphlet entitled "The Physical Bases of Ballistic Table Computations" is shortly to appear as an Ordnance Pamphlet, in the series "Notes on the Construction of Ordnance." This pamphlet by Professor Bennett involves no mathematical computations on the part of the reader and is rather a critical résumé of the latest practices in ballistic theory in so far as they deal with the topic mentioned. It is designed for use as a reference text in courses on gunnery, and will also constitute a part of the general introduction to the ballistic tables now being prepared under the supervision of the author.

14. In this paper Professor Bennett discusses the elementary question of the definition of distance in analytical geometry. The distance between points and between a point and a line are assumed as defined up to the algebraic sign. Three simple conventions are possible, (1) the arithmetical, by means of which all distances are positive, (2) the transcendental, by which distances are measured vectorially with a convention arbitrarily fixed in each case, (3) the algebraic, in which either

sign is always possible. Each definition has its objections, the most common usage being perhaps the least convenient. The difficulties are illustrated by very simple examples. The paper appeared in the October number of the *American Mathematical Monthly.*

15. In his Columbia dissertation, published in the *Transactions* (1918), Dr. Simonds studied the order and the number of invariants of differential configurations; and in particular with respect to the group of all point transformations obtained results entirely different from those given by Rabut in his paper on "Invariants universelles" (1898), but consistent with Kasner's invariant of the second order (*American Journal of Mathematics,* 1906).

In the present paper, Dr. Simonds discusses the five types of infinite groups of point transformations and the three types for contact transformations, according to Lie's classification, finding for each type the smallest number of curves, with distinct tangents, through a common point, which have an invariant of any given order n. For example, under the conformal group, two curves obviously have an invariant of first order, four curves have one of second order (checking with a result given by Kasner), three curves have an invariant for each of the orders 3, 4, 5, $\cdots$. For each type of group the simplest invariants are given explicitly.

16. In his discussion of the general geometric properties of systems of dynamical trajectories, Professor Kasner incidentally obtained the simple theorem that any particle starting from rest in any positional field of force describes a trajectory whose initial curvature is one third the curvature of the line of force passing through the starting point (*Transactions,* 1905, Princeton Colloquium Lectures 1913, page 9). This result holds for space of any dimensionality. In the present paper the theorem is extended to the motion of any number of particles acted upon by any positional forces, conservative or non-conservative. The ratio 1:3 remains valid even in the case of a resisting medium provided the resistance approaches zero when the velocity approaches zero.

F. N. COLE,
Secretary.