in the following journals: *Aviation*, *Aeronautics*, *Aeronautical Journal*, *Aerial Age*, *Proceedings of the Royal Society of London*, volume 91A, page 354, page 503, and *Aviation and Aeronautical Engineering* (which gave two lists of books on aviation, August 15, 1916, page 31, May 1, 1917, page 306).

A PARTIAL ISOMORPH OF TRIGONOMETRY.

BY PROFESSOR E. T. BELL.

1. It is well known that the only continuous solution, $\varphi(x), \psi(x)$, of the system of functional equations

$$\varphi(a+b) = \varphi(a)\varphi(b) - \psi(a)\psi(b), \tag{1}$$

$$\psi(a+b) = \psi(a)\varphi(b) + \varphi(a)\psi(b), \tag{2}$$

$$\varphi^2(a) + \psi^2(a) = 1 \tag{3}$$

is $\varphi(x) \equiv \cos x$, $\psi(x) \equiv \sin x$. By suppressing the condition that φ, ψ, shall be continuous functions of a single variable, and one or two of (1), (2), (3), we get what may be called the partial isomorphs of trigonometry, whose interest, of course, will depend chiefly upon their interpretations. Several such are known and in use. While seeking arithmetical paraphrases for some of the more complicated results in elliptic and theta functions, I noticed incidentally another of these isomorphs in which (1), (2) only are retained. Apart from its usefulness in the theory of numbers (which is not considered here), this isomorph is of interest because it extends, in a sense, the concepts of evenness and oddness, or parity, relatively to functions of several variables.

2. We denote the sets of variables, $(x_1, x_2, \cdots, x_n)$, $(-x_1, -x_2, \cdots, -x_n)$ by ξ, $-\xi$ respectively, and define a function f to be even or odd in ξ according as it does not or does change sign when the signs of $x_1, x_2, \cdots, x_n$ are changed simultaneously: $f(\xi) = f(-\xi)$, or $f(\xi) = -f(-\xi)$, according as f is even or odd in ξ. These relations may be written $f((x_1, x_2, \cdots, x_n)) = \pm f((-x_1, -x_2, \cdots, -x_n))$, the inner ( ) being used to distinguish $f(\xi)$ from $f(x_1, x_2, \cdots, x_n)$, in which no property of evenness or oddness is implied.

Two sets of variables ξ', ξ'', are distinct if and only if they have no variable in common. Unless the contrary is expressly stated, all the sets of variables in the f's which follow are distinct. We shall be concerned with a form of addition for sets, which is as follows: Let $\xi' \equiv (x_1', x_2', \cdots, x_a')$, $\xi'' \equiv (x_1'', x_2'', \cdots, x_b'')$, $\cdots$, $\xi''' \equiv (x_1''', x_2''', \cdots, x_c''')$; then the sum of $\xi', \xi'', \cdots, \xi'''$ is the set $(\xi'; \xi''; \cdots; \xi''')$, $\equiv (x_1', x_2', \cdots, x_a', x_1'', x_2'', \cdots, x_b'', \cdots, x_1''', x_2''', \cdots, x_c''')$. Hence in particular, $\xi' \equiv (x_1'; x_2'; \cdots; x_a') \equiv (x_1', x_2', \cdots, x_a')$; and $(\xi'; (\xi''; \xi'''); \cdots) \equiv (\xi'; \xi''; \xi'''; \cdots)$ etc.; and $(\xi'; -\xi''; \xi'''; \cdots) \equiv -(-\xi'; \xi''; -\xi'''; \cdots)$.

3. Extending the foregoing to several sets, let $(x_{i1}, x_{i2}, x_{i3}, \cdots, x_{ia_i}) \equiv \xi_i$; $(y_{j1}, y_{j2}, y_{j3}, \cdots, y_{jb_j}) \equiv \eta_j$, $(i = 1, \cdots, r;$ $j = 1, \cdots, s)$, denote $(r + s)$ distinct sets of variables. Then, a function f of $\xi_1, \xi_2, \cdots, \xi_r, \eta_1, \eta_2, \cdots, \eta_s$ which is even in each ξ separately, odd in each η separately, is denoted by $f(\xi_1, \xi_2, \cdots, \xi_r | \eta_1, \eta_2, \cdots, \eta_s)$. By definition the parity of this f is $(a_1, a_2, \cdots, a_r | b_1, b_2, \cdots, b_s)$; its order, or total number of independent x, y variables, is $(a_1 + a_2 + \cdots + a_r) + (b_1 + b_2 + \cdots + b_s)$; its degree, or total number of ξ, η variables is $(r + s)$. Similarly $f(\xi_1, \xi_2, \cdots, \xi_r | 0)$, $f(0 | \eta_1, \eta_2, \cdots, \eta_s)$ denote f's of respective parities $(a_1, a_2, \cdots, a_r | 0)$, $(0 | b_1, b_2, \cdots, b_s)$, of the obviously corresponding orders and degrees, even in each ξ, odd in each η. Beyond the prescribed conditions of evenness and oddness, all of these f's are wholly arbitrary: they may be continuous in some or in none of the x, y variables; algebraic, or transcendental, etc. When $a_i = a$, $b_j = b$, $(i = 1, \cdots, r; j = 1, \cdots, s)$, the above parities are written respectively $(a^r | b^s)$, $(a^r | 0)$, $(0 | b^s)$. The parity of a constant, i. e., function independent of the x, y variables in f, is $(0 | 0)$. According to this notation, a function of parity $(1^r | 1^s)$ is a function of $r + s$ single variables (not sets), even in r of the variables separately, odd separately in the rest; e. g.,

$$\prod_{i=1}^{r} \cos x_i \cdot \prod_{j=1}^{s} \sin y_j$$

is of parity $(1^r | 1^s)$; while

$$\cos\left(\sum_{i=1}^{r} x_i\right) \cdot \sin\left(\sum_{j=1}^{s} y_j\right)$$

is of parity $(r | s)$. As another example, an arbitrary function of n variables can always be written as the sum of a function of parity $(n|0)$ and a function of parity $(0|n)$:

$$(4)\quad 2f(x_1, x_2, \cdots, x_n) \equiv f_0((x_1, x_2, \cdots, x_n)|0) + f_1(0|(x_1, x_2, \cdots, x_n))$$

$$(5)\quad f_0((x_1, x_2, \cdots, x_n)|0) = f(x_1, x_2, \cdots, x_n) + f(-x_1, -x_2, \cdots, -x_n)$$

$$f_1(0|(x_1, x_2, \cdots, x_n)) = f(x_1, x_2, \cdots, x_n) - f(-x_1, -x_2, \cdots, -x_n).$$

The isomorph we shall consider is the algebra of sets and parities, and the closely related properties of functions $f(\xi_1, \xi_2, \cdots, \xi_r|\eta_1, \eta_2, \cdots, \eta_s)$. The algebra is implicit in § 5; the correspondence with (1), (2) is established in § 6, extended in §§ 7, 8, and in §§ 9, 10 outlined for the purpose stated in § 4.

4. It is a fundamental result for the applications that an arbitrary f of parity $(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$ may be expressed linearly in terms of $2^{\omega-\delta}$ suitably chosen functions whose parities are all of the type $(1^\alpha|1^\beta)$, where $\omega =$ the order of f; $\delta =$ the degree; and $\alpha + \beta = \omega$. Although the actual linear representation is not required in use, we shall, nevertheless, show how to write it down, in proving its existence—to bring out the trigonometric analogies between $\sin x$, $\cos x$ and $(0|x)$, $(x|0)$. It will follow then that the f in (4) is a linear function of the same sort; in this case the number of functions in the representation is 2^n, and $\alpha + \beta = n$. The degrees of the several functions in the representations follow a more complicated law, which may be written down if desired. To arrive at these representations briefly, we adopt the following conventions.

5. Associated with each symbol $(\xi_1, \xi_2, \cdots, \xi_r|\eta_1, \eta_2, \cdots, \eta_s)$ is a symbol of parity $(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$; and if $\xi_1 \equiv (\xi'; \xi''; \cdots; \xi''')$, the notation being that of § 2, the parity is written $((a; b; \cdots; c), a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$; viz., $a_1 \equiv (a; b; \cdots; c)$, and so in all cases where a set ξ or η is regarded as a sum of sets. With the same notation, if $(\xi'; \xi''; \cdots; \xi''')$ and $(a; b; \cdots; c)$ are associates, then we take as the associate of $(-\xi'; -\xi''; \cdots; \xi''')$, the parity $(-a; -b; \cdots; c)$, etc., the minus signs indicating that the signs of all the variables in $\xi', \xi'', \cdots$ have been changed. Hence, from the

definitions, we take $-(-a; -b; \cdots; c) \equiv (a; b; \cdots; -c)$, etc. For brevity we may write a symbol $(\xi_1, \xi_2, \cdots, \xi_r | \eta_1, \eta_2, \cdots, \eta_s) \equiv (X | Y)$, and the corresponding parity $(A | B)$. Then the formal laws of addition and multiplication for symbols $(X | Y)$, $(A | B)$ are defined as follows. An equation of the form

$$(6) \quad p(A | B) = c_1 p(A_1 | B_1) + c_2 p(A_2 | B_2) + \cdots + c_k p(A_k | B_k),$$

where the $c_1, c_2, \cdots, c_k$ are integers > 0, is to be read: An arbitrary f of parity $(A | B)$ is expressible as a sum of $c_1 + c_2 + \cdots + c_k$ suitably chosen functions, c_i of which are of parity $(A_i | B_i)$, $(i = 1, \cdots, k)$. Similarly, if some of the c's are integers < 0, (6) signifies the like expressibility as a difference. When we wish to consider the sets, rather than the parities, the equation (6) will be replaced by

$$(7) \quad q(X | Y) = c_1 q(X_1 | Y_1) + c_2 q(X_2 | Y_2) + \cdots + c_l q(X_l | Y_l),$$

where all the c's denote $+1$ or -1 (for an arbitrary constant, $|c| > 1$, may obviously be absorbed in the function). (7) indicates the appropriate $(X_i | Y_i)$ for the linear expression of arbitrary $f(X | Y)$; the equations (6), (7) are merely different ways of stating the same fact. The next has another meaning. If a given $F(X | Y)$ may be written in the form

$$(8) \qquad cF(X | Y) = c_1 f(X_1 | Y_1) + c_2 f(X_2 | Y_2) + \cdots + c_k f(X_k | Y_k),$$

where the c's are constants, by choosing the ξ, η which occur in X_i, Y_i, $(i = 1, \cdots, k)$, suitably in terms of the ξ, η which occur in X, Y this will be symbolized by (9), equivalent to (8),

$$(9) \quad cq(X | Y) = [c_1 q(X_1 | Y_1) + c_2 q(X_2 | Y_2) + \cdots + c_k q(X_k | Y_k)].$$

To define multiplication, let $t, \alpha, \beta \equiv p, A, B$, or $t, \alpha, \beta \equiv q, X, Y$; then multiplication, which is defined to be associative, commutative and distributive, for symbols $t(\alpha | \beta)$, is given by

$$(10)\quad t(\alpha_1|\beta_1)t(\alpha_2|\beta_2)\cdots t(\alpha_r|\beta_r) = t(\alpha_1, \alpha_2, \cdots, \alpha_r|\beta_1, \beta_2, \cdots, \beta_r)$$

$$(11)\quad t(\alpha|\beta)\{t(\alpha_1|\beta_1) + t(\alpha_2|\beta_2)\} = t(\alpha, \alpha_1|\beta, \beta_1) + t(\alpha, \alpha_2|\beta, \beta_2),$$

$$t(-\alpha|\beta) = t(\alpha|\beta) = -t(\alpha|-\beta).$$

The identity of addition is 0, of multiplication $(0|0)$. These laws are to be regarded as purely formal until they receive their interpretations in the addition theorems for $t(\alpha|\beta)$. One obvious interpretation of (10) is: The product of r functions on the distinct sets $(X_i|Y_i)$, $(i = 1, \cdots, r)$, is a function of parity $(A_1, A_2, \cdots, A_r|B_1, B_2, \cdots, B_r)$. Similarly for (11); but both of these are trivial. We may now write down the equations which establish the correspondence with (1), (2).

6. Let $t, \alpha, \beta \equiv p, a, b$; or $t, \alpha, \beta \equiv q, \xi', \xi''$, where $\xi' = (x_1', x_2', \cdots, x_a')$, $\xi'' = (x_1'', x_2'', \cdots, x_b'')$; then the addition theorems for t-symbols are

$$(12)\quad t((\alpha; \beta)|0) = t(\alpha|0)t(\beta|0) - t(0|\alpha)t(0|\beta), \equiv t(\alpha, \beta|0) - t(0|\alpha, \beta),$$

$$(13)\quad t(0|(\alpha; \beta)) = t(0|\alpha)t(\beta|0) + t(\alpha|0)t(0|\beta), \equiv t(\beta|\alpha) + t(\alpha|\beta),$$

the second forms coming from (10). Corresponding to the formulas for $\sin\alpha\cos\beta$, etc., we have

$$(14)\quad 2t(\alpha|0)t(\beta|0) \equiv 2t(\alpha, \beta|0) = [t((\alpha; -\beta)|0) + t((\alpha; \beta)|0)],$$

$$(15)\quad 2t(0|\alpha)t(0|\beta) \equiv 2t(0|\alpha, \beta) = [t((\alpha; -\beta)|0) - t((\alpha; \beta)|0)],$$

$$(16)\quad 2t(0|\alpha)t(\beta|0) \equiv 2t(\beta|\alpha) = [t(0|(\alpha; \beta)) + t(0|(\alpha; -\beta))],$$

$$(17)\quad 2t(\alpha|0)t(0|\beta) \equiv 2t(\alpha|\beta) = [t(0|(\alpha; \beta)) - t(0|(\alpha; -\beta))].$$

(16), (17) are, of course, different forms of the same thing, but it is convenient to have both.

To prove (12)–(17), we write down the identities (18) for (12), (14), (15), and (19) for (13), (16), (17),

$$f((\xi'; \xi'')|0) = f_1(\xi', \xi''|0) - f_2(0|\xi', \xi''),$$

$$(18)\quad 2f_1(\xi', \xi''|0) \equiv f((\xi'; -\xi'')|0) + f((\xi'; \xi'')|0),$$

$$2f_2(0|\xi', \xi'') \equiv f((\xi'; -\xi'')|0) - f((\xi'; \xi'')|0).$$

$$
\begin{aligned}
&f(0\,|\,(\xi';\xi'')) = f_1(\xi''\,|\,\xi') + f_2(\xi'\,|\,\xi''),\\
(19)\qquad &2f_1(\xi''\,|\,\xi') \equiv f(0\,|\,(\xi';\xi'')) + f(0\,|\,(\xi';-\xi'')),\\
&2f_2(\xi'\,|\,\xi'') \equiv f(0\,|\,(\xi';\xi'')) - f(0\,|\,(\xi';-\xi'')).
\end{aligned}
$$

It only remains to show that the functions on the right of the f_1, f_2 identities have the properties implied by the notation on the left, thus, e. g.,

$$
\begin{aligned}
2f_1(-\xi''\,|\,\xi') &\equiv f(0\,|\,(\xi';-\xi'')) + f(0\,|\,(\xi';\xi'')) = 2f_1(\xi''\,|\,\xi');\\
2f_1(\xi''\,|-\xi') &\equiv f(0\,|\,(-\xi';\xi'')) + f(0\,|\,(-\xi';-\xi''))\\
&= -f(0\,|-(-\xi';\xi'')) - f(0\,|-(-\xi';-\xi''))\\
&= -f(0\,|\,(\xi';-\xi''))-f(0\,|\,(\xi';\xi''))=-2f_1(\xi''\,|\,\xi');
\end{aligned}
$$

and similarly the others may be verified.

7. Comparing (12), (13) with (1), (2), we see that a correspondence has been established such that if in any consequence of (1), (2) alone, when written, $\cos(a+b) = \cos a \cos b - \sin a \sin b$, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, $\cos *$, $\sin *$ be replaced respectively by $t(*\,|\,0)$, $t(0\,|\,*)$, we get a result which by (12), (13) is true, and which may be translated at once into terms of functions f. Thus the formulas for the sine or cosine of n angles as a sum of products of sines and cosines of the several angles are consequences of (1), (2) alone; hence they may be translated as indicated. E. g., corresponding to $\cos(a+b+c) =$ etc.; we have

$$
\begin{aligned}
t((\alpha;\beta;\gamma)\,|\,0) &= t(\alpha\,|\,0)t(\beta\,|\,0)t(\gamma\,|\,0) - t(\alpha\,|\,0)t(0\,|\,\beta)t(0\,|\,\gamma)\\
&\qquad - t(\beta\,|\,0)t(0\,|\,\gamma)t(0\,|\,\alpha) - t(\gamma\,|\,0)t(0\,|\,\alpha)t(0\,|\,\beta)\\
&\equiv t(\alpha,\beta,\gamma\,|\,0) - t(\alpha\,|\,\beta,\gamma) - t(\beta\,|\,\gamma,\alpha) - t(\gamma\,|\,\alpha,\beta),
\end{aligned}
$$

which asserts that f_0, f_1, f_2, f_3 may be found such that $f((\xi_1;\xi_2;\xi_3)\,|\,0) \equiv f_0(\xi_1,\xi_2,\xi_3\,|\,0) - f_1(\xi_1\,|\,\xi_2,\xi_3) - f_2(\xi_2\,|\,\xi_3,\xi_1) - f_3(\xi_3\,|\,\xi_2,\xi_1)$. The actual forms of f_0, f_1, f_2, f_3 may be written down as indicated in § 9, or they follow from (14)–(17) on multiplying throughout by $t(0\,|\,\gamma)$, $t(\gamma\,|\,0)$ and reapplying (14) to (17) to reduce terms of the form $t(\gamma\,|\,(\alpha;\pm\beta))$, $t((\alpha;\pm\beta)\,|\,\gamma)$, $t((\alpha;\pm\beta),\gamma\,|\,0)$, etc.; they are

$$
\begin{aligned}
4f_0(\xi_1, \xi_2, \xi_3 | 0) &\equiv \quad f + f' + f'' + f''', \\
4f_1(\xi_1 | \xi_2, \xi_3) &\equiv - f - f' + f'' + f''', \\
4f_2(\xi_2 | \xi_3, \xi_1) &\equiv - f + f' - f'' + f''', \\
4f_3(\xi_3 | \xi_1, \xi_2) &\equiv - f + f' + f'' - f''',
\end{aligned}
$$

where

$$
\begin{aligned}
f &\equiv f((\ \xi_1;\ \ \xi_2;\ \ \xi_3) | 0), \\
f' &\equiv f((- \xi_1;\ \ \xi_2;\ \ \xi_3) | 0), \\
f'' &\equiv f((\ \xi_1; - \xi_2;\ \ \xi_3) | 0), \\
f''' &\equiv f((\ \xi_1;\ \ \xi_2; - \xi_3) | 0).
\end{aligned}
$$

Similarly, corresponding to the development of sin $(a + b + c)$,

$$
\begin{aligned}
t(0 | (\alpha; \beta; \gamma)) = t(\beta, \gamma | \alpha) + t(\gamma, \alpha | \beta) \\
+ t(\alpha, \beta | \gamma) - t(0 | \alpha, \beta, \gamma);
\end{aligned}
$$

$$
\begin{aligned}
g(0 | (\xi_1; \xi_2; \xi_2)) \equiv g_1(\xi_2, \xi_3 | \xi_1) + g_2(\xi_3, \xi_1 | \xi_2) \\
+ g_2(\xi_1, \xi_2 | \xi_3) - g_0(0 | \xi_1, \xi_2, \xi_3):
\end{aligned}
$$

$$
\begin{aligned}
4g_1(\xi_2, \xi_3 | \xi_1) &\equiv \quad g - g' + g'' + g''', \\
4g_2(\xi_3, \xi_1 | \xi_2) &\equiv \quad g + g' - g'' + g''', \\
4g_3(\xi_2, \xi_1 | \xi_3) &\equiv \quad g + g' + g'' - g''', \\
4g_0(0 | \xi_1, \xi_2, \xi_3) &\equiv - g + g' + g'' + g''',
\end{aligned}
$$

where

$$
\begin{aligned}
g &\equiv g(0 | (\ \xi_1;\ \ \xi_2;\ \ \xi_3)), \\
g' &\equiv g(0 | (- \xi_1;\ \ \xi_2;\ \ \xi_3)), \\
g'' &\equiv g(0 | (\ \xi_1; - \xi_2;\ \ \xi_3)), \\
g''' &\equiv g(0 | (\ \xi_1;\ \ \xi_2; - \xi_3)).
\end{aligned}
$$

As a last example, combining these, we will write $f(x_1, x_2, x_3)$, f arbitrary, as a linear function of 8 functions of parities $(1^\alpha | 1^\beta)$, $\alpha + \beta = 3$, as stated in § 4.

$$
2f(x_1, x_2, x_3) = F((x_1, x_2, x_3) | 0) + G(0 | (x_1, x_2, x_3));
$$

$$
\begin{cases}
F((x_1, x_2, x_3) | 0) = f(x_1, x_2, x_3) + f(- x_1, - x_2, - x_3), \\
G(0 | (x_1, x_2, x_3)) \equiv f(x_1, x_2, x_3) - f(- x_1, - x_2, - x_3);
\end{cases}
$$

$$\begin{cases} F((x_1, x_2, x_3)\,|\,0) = F_0(x_1, x_2, x_3\,|\,0) - F_1(x_1\,|\,x_2, x_3) \\ \qquad\qquad - F_2(x_2\,|\,x_1, x_3) - F_3(x_3\,|\,x_1, x_2), \\ G(0\,|\,(x_1, x_2, x_3)) = G_1(x_2, x_3\,|\,x_1) + G_2(x_3, x_1\,|\,x_2) \\ \qquad\qquad + G_3(x_1, x_2\,|\,x_3) - G_0(0\,|\,x_1, x_2, x_3). \end{cases}$$

Whence, on applying the foregoing expansions for $t((\alpha; \beta; \gamma)\,|\,0)$, $t(0\,|\,(\alpha; \beta; \gamma))$ to F, G, we find, on putting

$$\begin{array}{ll} f_0 \equiv f(\ \ x_1,\ \ x_2,\ \ x_3), & f_4 = f(\ \ x_1, -x_2,\ \ x_3), \\ f_1 \equiv f(-x_1, -x_2, -x_3), & f_5 \equiv f(-x_1,\ \ x_2, -x_3), \\ f_2 \equiv f(-x_1,\ \ x_2,\ \ x_3), & f_6 \equiv f(\ \ x_1,\ \ x_2, -x_3), \\ f_3 \equiv f(\ \ x_1, -x_2, -x_3), & f_7 \equiv f(-x_1, -x_2,\ \ x_3), \end{array}$$

the following values for F_0, F_1, F_2, F_3, G_1, G_2, G_3, G_0:

$$\begin{array}{lll} 4F_0(x_1, x_2, x_3\,|\,0) & = & f_0 + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 \\ 4F_1(x_1\,|\,x_2, x_3) & = & -f_0 - f_1 - f_2 - f_3 + f_4 + f_5 + f_6 + f_7 \\ 4F_2(x_2\,|\,x_3, x_1) & = & -f_0 - f_1 + f_2 + f_3 - f_4 - f_5 + f_6 + f_7 \\ 4F_3(x_3\,|\,x_1, x_2) & = & -f_0 - f_1 + f_2 + f_3 + f_4 + f_5 - f_6 - f_7 \\ 4G_1(x_2, x_3\,|\,x_1) & = & f_0 - f_1 - f_2 + f_3 + f_4 - f_5 + f_6 - f_7 \\ 4G_2(x_3, x_1\,|\,x_2) & = & f_0 - f_1 + f_2 - f_3 - f_4 + f_5 + f_6 - f_7 \\ 4G_3(x_1, x_2\,|\,x_3) & = & f_0 - f_1 + f_2 - f_3 + f_4 - f_5 - f_6 + f_7 \\ 4G_0(0\,|\,x_1, x_2, x_3) & = & -f_0 + f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - f_7 \end{array}$$

As a check, we find from these values

$$8f_0 \equiv 8f(x_1, x_2, x_3) = 4(F_0 - F_1 - F_2 - F_3 + G_1 + G_2 + G_3 - G_0).$$

It thus appears from (12), (13) that the semicolon in $(\alpha; \beta)$ plays the same part as the plus sign in $(a + b)$ in (1), (2). We next interpret the $|$ and the commas in a symbol $t((\alpha; \beta; \cdots), \gamma, \cdots\,|\,(\alpha'; \beta'; \cdots), \delta, \cdots)$.

8. To keep the writing simple, we may take the case of $t((\alpha; \beta), \gamma\,|\,0)$; the general case is treated in precisely the same way. Multiplying (12), (14), (15) throughout by $t(\gamma\,|\,0)$, we get for (12)

$$(20) \quad t((\alpha; \beta), \gamma\,|\,0) = [t(\alpha\,|\,0)t(\beta\,|\,0) - t(0\,|\,\alpha)t(0\,|\,\beta)]t(\gamma\,|\,0),$$
$$\equiv t(\alpha, \beta, \gamma\,|\,0) - t(\gamma\,|\,\alpha, \beta).$$

Assuming for the moment that this is a true equation, we see that the comma in $t((\alpha;\beta),\gamma|0)$ corresponds to the $\times$ sign in $\cos(\alpha+\beta)\times\cos\gamma=[\cos\alpha\cos\beta-\sin\alpha\sin\beta]\times\cos\gamma$ $=$ etc.; and likewise in the general case for multiplication by $t(\gamma_1,\gamma_2,\cdots,\gamma_r|\delta_1,\delta_2,\cdots,\delta_s)$, the commas play the part of $\times$ signs. To see that (20) is a true equation, we remark that it is included in (12), (14), (15), when, considering $F((\xi';\xi''),\xi'''|0)$ as a function of $(\xi';\xi'')$ alone, we write $F((\xi';\xi''),\xi'''|0)\equiv f((\xi';\xi'')|0)$, and set in (18)

$$f_1(\xi',\xi''|0)\equiv F_1(\xi',\xi'',\xi'''|0)\quad f_2(0|\xi',\xi'')\equiv F_2(\xi'''|\xi',\xi''),$$

whence the second and third of (18) become:

$$\begin{cases} 2F_1(\xi',\xi'',\xi'''|0)\equiv F((\xi';-\xi''),\xi'''|0)+F((\xi';\xi''),\xi'''|0), \\ 2F_2(\xi'''|\xi',\xi'')\quad\equiv F((\xi';-\xi''),\xi'''|0)-F((\xi';\xi''),\xi'''|0); \end{cases}$$

corresponding to (14), (15) multiplied by $t(\gamma|0)$:

$$\begin{cases} 2t(\alpha|0)t(\beta|0)t(\gamma|0)\equiv 2t(\alpha,\beta,\gamma|0) \\ \qquad\qquad = [t((\alpha;-\beta),\gamma|0)+t((\alpha;\beta),\gamma|0)], \\ 2t(0|\alpha)t(0|\beta)t(\gamma|0)\equiv 2t(\gamma|\alpha,\beta) \\ \qquad\qquad = [t((\alpha;-\beta),\gamma|0)-t((\alpha;-\beta),\gamma|0)]. \end{cases}$$

Similarly, if (12) is multiplied by $t(0|\gamma)$, it is easily seen that the $|$ is a symbolic $\times$.

9. (i) Combining the correspondences established in §§ 7, 8, we have the following correspondence between t-symbols and sines and cosines. Let

$$\alpha_i\equiv(\alpha_{i1};\alpha_{i2};\cdots;\alpha_{im_i}),\quad \beta_j\equiv(\beta_{j1};\beta_{j2};\cdots;\beta_{jn_j}),$$
$$(i=1,\cdots,r;j=1,\cdots,s),$$

be any partitioning of the α_i, β_j in $t\equiv t(\alpha_1,\alpha_2,\cdots,\alpha_r|\beta_1,\beta_2,\cdots,\beta_s)$. Then, as the generalization of (12), (13), t may be expanded as a linear function, with coefficients ± 1, of homogeneous products all of whose factors are $t(*|0)$'s or $t(0|\S)$'s by expanding

$$\prod_{i=1}^{r}\cos\left(\sum_{a=1}^{m_i}\alpha_{ia}\right)\cdot\prod_{j=1}^{s}\sin\left(\sum_{b=1}^{n_j}\beta_{jb}\right)$$

in the usual form as a sum of homogeneous products of cos $*$'s sin §'s, each $*$, § representing a single α or a single β, and then replacing cos $*$, sin § respectively by $t(*\,|\,0)$, $t(0\,|\,\S)$. Finally, applying (10) to each term in the t-expansion, we reduce the whole to a $\Sigma \pm t(\alpha', \alpha'', \cdots | \beta', \beta'', \cdots)$, the α', α'', $\cdots$, β', β'', $\cdots$ being the α, β in the ( ) for the partitioning of the α_i, β_j.

(ii) Similarly, the generalization of (14), viz., the expression of $2^{s-1}t(\alpha_1|0)t(\alpha_2|0) \cdots t(\alpha_s|0)$ in the form $[\Sigma \pm t(\pm \alpha_1; \pm \alpha_2; \cdots; \pm \alpha_s)|0)]$, is written down from the expression of $2^{s-1}\cos \alpha_1 \cos \alpha_2 \cdots \cos \alpha_r$ in the form $\Sigma \pm \cos(\alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_s)$, a cosine term in the Σ, such as $\cos(\alpha_1 - \alpha_2 + \cdots + \alpha_s)$ being replaced by $t(\alpha_1; -\alpha_2; \cdots; \alpha_s)$; likewise for $c \cdot t(0|\beta_1)$ $t(0|\beta_2) \cdots t(0|\beta_s)$, where $c = (-1)^{s/2}2^{s-1}$ or $(-1)^{(s-1)/2}2^{s-1}$ according as s is even or odd, from the corresponding cosine or sine sums for $c \sin \beta_1 \sin \beta_2 \cdots \sin \beta_s$. It is unnecessary to write the results: they may be obtained at once from the formulas given in books on trigonometry, e. g., E. W. Hobson's Elementary Treatise, §§ 49, 50 (Cambridge, 1897).

(iii) To generalize (16), (17), we multiply together the expressions found in (ii) for $t(\alpha_1|0)t(\alpha_2|0) \cdots t(\alpha_r|0)$ and $t(0|\beta_1)t(0|\beta_2) \cdots t(0|\beta_s)$, using (10) on each term of the distributed product. There will be four cases, according as $r, s \equiv 0, 1 \bmod 2$. The results are rather complicated, and need not be written.

(iv) Using (i)–(iii) it is evident that $f(\xi_1, \xi_2, \cdots, \xi_r | \eta_1, \eta_2, \cdots, \eta_s)$ may be systematically expressed in the form stated in § 4. We have given a simple example in § 7. All of the results of this section may be proved independently by induction. This is unnecessary, however, as the formal correspondence established in §§ 7, 8 shows.

10. On account of its interest, we may conclude with the analogue of Demoivre's theorem. Since this theorem, for a positive integral exponent n, follows from (1), (2) alone, we may apply the correspondence of § 7 to get $p(n|0)$ and $p(0|n)$:

$$\{p(1|0) + ip(0|1)\}^n = p(n|0) + ip(0|n),$$

whence, equating reals and imaginaries after expansion of the left member,

$$(21)\quad \begin{cases} p(n\,|\,0) = p(1^n\,|\,0) - \binom{n}{2}p(1^{n-2}\,|\,1^2) \\ \qquad\qquad\qquad\qquad + \binom{n}{4}p(1^{n-4}\,|\,1^4) - \cdots, \\ p(0\,|\,n) = \binom{n}{1}p(1^{n-1}\,|\,1) - \binom{n}{3}p(1^{n-3}\,|\,1^3) \\ \qquad\qquad\qquad\qquad + \binom{n}{5}p(1^{n-5}\,|\,1^5) - \cdots. \end{cases}$$

The q-forms of these are found in the same way by distributing the product, and equating, in

$$(22)\quad \prod_{j=1}^{n}\{q(\xi_j\,|\,0) + iq(0\,|\,\xi_j)\} \\ = q((\xi_1, \xi_2, \cdots, \xi_n)\,|\,0) + iq(0\,|\,(\xi_1, \xi_2, \cdots, \xi_n)).$$

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THE TRAINS FOR THE 36 GROUPLESS TRIAD SYSTEMS ON 15 ELEMENTS.

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In the *Transactions* for January, 1913, Professor H. S. White derived a new method for the comparison of triad systems, and applied it to the two triad systems on 13 elements. One part of a memoir published in the *Memoirs of the National Academy of Sciences*, volume 14, second memoir, shows the efficacy of this method in determining the groups, and in establishing the noncongruency of the 44 systems Δ_{15} which have a group different from the identity. The present paper gives the results of this method of comparison applied to the 36 groupless systems Δ_{15}. In this method, the triad system is regarded as an operator, and certain covariants of that operator are deduced. These covariants can be represented graphically and are called the trains of the system.