MATHEMATICS IN WAR PERSPECTIVE.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 27, 1918.

BY PRESIDENT L. E. DICKSON.

AN army officer of high rank, now facing the problems involved in stopping the huge war machine which he had helped to build, recently remarked to me that this getting out of war is far more trouble than getting into it. The armistice has put me in the same boat and, for the purposes of this address, came a few weeks too soon. I had already put myself under obligations to numerous friends, including two in England and France, for furnishing me authoritative information on the rôle of mathematics and its applications in the war. While this information is fortunately no longer needed for its initial purpose, it bears on the timely question of the kind of preparedness which the nation should adopt. While science has played an important rôle in this war, it would undoubtedly play a dominant rôle in a future war, and no scheme of national preparedness will prove adequate which does not insure an ample supply of highly trained scientists and furnish to all men effective training in the fundamentals of exact science. Owing to its recognized value as a fundamental part of military education, I expressly include mathematics, especially trigonometry and graphical analysis. Let it not again become possible that thousands of young men shall be so seriously handicapped in their army and navy work by lack of adequate preparation in these subjects. Nor should so many instructors in courses for prospective officers again be chosen from those who had just passed hastily through the course, not to count those who had merely taken a few private lessons. Fortunately the more widespread and more effective scientific training here advocated as an essential part of national preparedness for war furnishes at the same time the surest means to retain and increase our material prosperity, to add to our health, comfort and conveniences, and so to train our youth in the unravelling of the mysteries of the universe and in habits of drawing accurate conclusions from correctly observed facts that they may the more surely become sane, reliable and efficient citizens.

For data on the military and naval instruction in France during the war, I am indebted to the distinguished mathematician, M. Edmond Maillet, President of the Mathematical Society of France, who also kindly sent me current programmes of requirements for admission to the various schools. As a background we need some facts concerning the instruction given just prior to the war. Cadets to become officers of infantry or cavalry took a two-year course (which was suspended during the period of the war) at the Ecole Spéciale Militaire de Saint-Cyr, the entrance examinations being on algebra (through quadratics), geometry, trigonometry, conic sections, notions of derivatives, descriptive geometry, mechanics, and general physics and chemistry. But the future officer of artillery or engineering was trained at other schools which required more extensive preparation for entrance. After being able to pass the entrance examinations at Saint-Cyr, he entered a special class at a lycée which continued 8 or 9 months prior to October, 1917, but only $5\frac{1}{2}$ months in 1917–18. For example, in the class of Mathématiques Spéciales, preparing particularly for the Ecole Polytechnique, the number of lessons, each of $1\frac{1}{2}$ to 2 hours, were as follows for the 8 (and $5\frac{1}{2}$) months: trigonometry, (college) algebra, and differential and integral calculus, 47 (36); plane and solid analytic geometry, 45 (33); descriptive geometry, 24 (22); mechanics, 20 (7); supplemented by written exercises, quizzes and 12 drawings in descriptive geometry. In such a lycée, he continued also his study of general physics and chemistry, history, and English or German.

The special aim of the famous Ecole Polytechnique is to provide the training in the exact sciences which is necessary for artillery officers and for the various types of engineers in their subsequent technical course at one of the various state schools of applied science, mentioned below. The subjects taught in the two-year course and the number of lectures are as follows:* higher analysis (65), projective, infinitesimal and cinematic geometry (26), application of descriptive geometry to stereotomy (12), mechanics (74), drawing (30), architecture (12), astronomy and geodesy (17), physics (60), chemistry (60), history and literature (40), political and social economy (20), German, English, military science (30), and drill (one hour a week).

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* *Journn. de l'Ecole Polyt.*, ser. 2, cah. 14, 1910, pp. I–XLIII.

Among the schools of applied science* are the Ecole des Ponts et Chaussées† for engineers of railroads, ports, rivers, harbors, drainage, etc.; Ecole du Génie Maritime for marine engineers; Ecole d'Hydrographie for a small number of hydrographic engineers, at each of which there is a two-year course primarily for those who have completed the course at the Ecole Polytechnique. The Ecole Nationale Supérieure des Mines trains especially mining engineers, but also for industrial positions; the engineers are taken exclusively from the Ecole Polytechnique and are given a three-year course. The Ecole Centrale des Arts et Manufactures offers a three-year course for engineers for all branches of industry, architecture, mining, machinery, chemistry, etc. The above schools are all at Paris.

The Ecole Navale at Brest offers to candidates of ages 16 to 19, who have had the equivalent of the above special lycée course (with omission of descriptive geometry and electricity in 1918), a two-year course for line officers in the navy. The subjects studied are analysis, rational mechanics, astronomy, navigation, naval architecture and machines, drawing, photography, physics, chemistry, literature, history and seamanship. The course is followed by a cruise of ten months for practical instruction. The entire course has been reduced to five months during the war. To provide further line officers in the navy, the Ecole des Elèves Officiers de Marine at Brest offers a two-year course for enlisted naval men of certain grades and lengths of service who can pass the examinations in arithmetic, (advanced high school) algebra, trigonometry, conic sections, notions of derivatives, elements of plane, solid, and descriptive geometry, mechanics, general physics, geography and French history.

To provide captains and mechanician officers for the merchant marine there are free state schools at 16 French ports, but only those at Havre, Nantes, Marseilles, and Paimpol were open during the war.‡ Beginning with 1918, the course at

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* P. Melon, L'Enseignement supérieur et L'Enseignement technique en France, ed. 2, 1893.

† Cf. transl. of French report by Major W. D. Connor, National School of Bridges and Highways, Paris, France. U. S. Printing Office, 1913, 42 pp.

‡ Organisation et Fonctionnement des Ecoles d'Hydrographie et de l'Institut Maritime du Havre; conditions d'admission, programmes des examens pour . . . capitaine au long cours, Paris, A. Challamel, 1918, 108 pp. Programme . . . d'officier mécanicien de la marine marchande, Paris, Vuibert, 68 pp.

the first three of these four schools will require two years. For the special "brevet supérieur" captain or mechanician, the examinations include also differential and integral calculus, rational mechanics, physics, elementary chemistry, machines, etc. There are schools for apprentice mechanicians at Lorient, Brest, Toulon, and Havre.

To supply the increased need for army officers during the war, the Ecoles Militaires d'Aspirants at Saint-Maixent, Saumur, Fontainbleau and Versailles for the infantry, cavalry, artillery and engineers, respectively, provided practical courses of five months for sub-officers, regarded as capable of becoming officers, who passed oral and written examinations in arithmetic, plane geometry, linear equations in several unknowns, definitions of the trigonometric functions and of the terms in solid geometry, formulas for surfaces and volumes (without proofs), elementary notions in physics and chemistry, geography, French history and literature. Also an extensive acquaintance with descriptive geometry was required of candidates for entrance to the school for engineers. No examination was required in the case of sub-officers who had served 15 months in the army and were recommended by the military authorities, nor of those who had been admitted to the Ecole Polytechnique or the school at Saint-Cyr.

In England the training of naval cadets under the system* adopted in 1913 consisted of a two-year course at the Royal Naval College at Osborne, followed by a two-year course at the Royal Naval College at Dartmouth, and six months on a training cruiser. In 1912 there were 439 cadets at Osborne and 406 at Dartmouth. To enter Osborne the candidate must be between $12\frac{2}{3}$ and 13 years of age and pass an entrance examination in arithmetic, algebra (linear equations in one or more unknowns), geometrical constructions and the substance of the first book of Euclid, history, geography, and languages. In arithmetic, algebra, geometry and the elements of plane trigonometry, there are $6\frac{3}{4}$ hours per week of instruction and 2 of preparation during the first four terms, each of twelve weeks, and $7\frac{1}{2} + 1\frac{1}{4}$ during the last two terms. At Dartmouth all cadets take algebra, plane and solid geometry and plane and spherical trigonometry, while the more proficient men take also analytic geometry and elementary notions of

* Great Britain Admiralty Committee on education and training of naval officers, Accounts and Papers, Navy, vol. 43, 1913, 170 pp.

calculus. To insure that all shall attain to the standard in navigation, extra time is provided in navigation for the weaker cadets at the expense of their further progress in mathematics. In the first two terms there are $5 + 2\frac{1}{2}$ hours of mathematics and no navigation, while the later schedules are

	3d term	4th term	5th term	6th term
Math.	$4+2$	$4+2\frac{1}{2}$ or $3\ +2$	$4+2\frac{1}{2}$ or $3+2$	$4+2\frac{1}{2}$ or $3+2$ or $2+1\frac{1}{2}$
Nav.	$1+\frac{1}{2}$	$2+1$ or $1\frac{1}{2}+\frac{1}{2}$	$2+1$ or $3+1\frac{1}{2}$	$2+1$ or $3+\frac{1}{2}$ or $4+2$

During the subsequent 24 weeks on a training cruiser, five hours were devoted daily to study; the time for the optional course in trigonometry and calculus was included in the six hours per week assigned to navigation.

Professor W. Burnside has kindly provided me with a statement of the instruction during the war. The only training in navigation for prospective officers of the British navy is given on board ship and at the Royal Naval College, Dartmouth. There the course was reduced to five terms by cutting down somewhat the non-professional subjects such as history. During the initial two terms, two hours per week were given to recitation and $\frac{1}{2}$ hour to preparation in navigation. In the third term the time ranged from $2 + 1$ hours for the best to $3 + 1\frac{1}{2}$ for the poorest of the six classes into which the cadets were separated on the basis of ability. In the fourth term the hours were $2 + 1\frac{1}{2}$ to $3 + 2$; in the final fifth term, $3 + 1\frac{1}{2}$ to $4 + 2$. The only text-book used is S. F. Card's Navigation Notes and Examples, 245 pages, second edition, 1917, Arnold (to be had of Longmans, Green and Company in America), but reference was made to Chapter XVII of the Admiralty Manual of Navigation, 525 pages, 1914, London (3 shillings). Cadets procured Inman's tables and the Nautical Almanac for 1919, and had access to Burdwood's Azimuth Tables. They acquired facility in using parallel rulers, dividers, sextants, magnetic and gyro compasses, station pointers, and the large scale chart S 389 D.

For artillery officers in the army and for gunnery officers in the navy, the whole of the theoretical and a great part of the practical training is given at the Ordnance College at Woolwich, where most of the courses are technical rather than theoretical. The only course which gives teaching on theoretical ballistics, external and internal, is one of 9 hours a week for

8 months for artillery officers who are practical experts of at least six years' experience in their profession. The books used are the Army and the Admiralty Gunnery Manuals which are strictly confidential documents. There are no public British text-books on ballistics.

The work at our own government schools* at West Point and Annapolis need not be reported on here. The emphasis during the first two years is on pure mathematics. There are various post-graduate army schools. In marine engineering the first year's post-graduate work is done at the Naval Academy and the second year at Columbia University. Other graduates, who are to become naval constructors, take a three-year course at the Massachusetts Institute of Technology. On account of the war, a class recently graduated at West Point after two years' work. Annapolis is temporarily on a three-year schedule, the enrollment in the entering, middle and graduating classes being now 963, 678 and 485 respectively, with 18 men per section in mathematics.

The Naval Auxiliary Reserve Training Schools at Pelham Bay and at the Municipal Pier of Chicago taught navigation, seamanship, etc., to a large number of enlisted men seeking an ensign's commission. As many men lacked adequate mathematical preparation for the work at the latter school, preliminary courses in trigonometry and navigation were given to about 900 of these men at the University of Chicago and to a like number at Northwestern University.

At the Officer Material School, Cambridge, Mass., the Navy conducted, during the past sixteen months, four-month courses in navigation (including plane and spherical trigonometry), ordnance and gunnery, seamanship, and naval regulations, the number of hours per week of class work being 8, 8, 7, and 2, respectively. The 1,000 students were selected from those enlisted in the navy and were all above 21 years of age. The instructors expressed belief in the need in the future of more thorough grounding in mathematics, up to and including trigonometry.

The U. S. Shipping Board conducted free schools for the

* Besides their annual registers, see the report of the International commission on the teaching of mathematics, U. S. Bureau of Education, Bulletin, 1912, No. 2; also, Report of U. S. Commissioner of Education, 1913, I, pp. 599–630. In the *Amer. Math. Monthly*, Oct., 1918, pp. 370–2, Professor Root described the temporary course of 16 weeks in navigation at Annapolis for reserve officers.

training of navigation and engineer officers for the merchant marine. The first schools were opened July 1, 1917, and others from time to time up to November, 1918,—31 schools in all. The total attendance has been 12,218; of these, 3,509 completed the course for deck officers and 3,290 the course for engineer officers. For the navigation schools the prerequisite was two years' sea experience and graduation from a grammar school. The instruction (30 hours per week in class and 10 of other study) was on navigation by dead reckoning and by observation (Bowditch) and in "Rules of the Road" (published by the Hydrographic Office), international code of signals, etc. Each school possessed six sextants, a chronometer, compass, azimuth instrument, and five dividers. In the schools for engineer officers the course of one month (36 hours a week) covered the technical knowledge required for the grade of Chief. The text-books were Dyson's Practical Marine Engineering, and the Crosby Company's Practical Instructions on the Steam Engine Indicator. More difficulty was found with mathematics than anything else and special instruction was given in mathematics in the early part of the course. On the average there was one instructor for ten students in these schools. For the preceding information I am indebted to Director Henry Howard and his assistants.

The Student Army Training Corps was formally inaugurated on October 1, 1918, at some 500 colleges and universities. By November 1 the enrollment had reached the following figures obtained from the War Department: army, collegiate section, 127,766; vocational section, 37,261; navy, 12,598; marine, 413; in process of induction, 976. The following conclusions are based upon 29 replies to a questionnaire from 17 large universities and 12 colleges and small universities, which together represent all parts of the country. At these schools, 14,785 men took trigonometry in 522 sections with an average of 28 men and four hours of class recitation. As 39 per cent. of the army and naval enrollment (exclusive of the vocational section) at these schools took trigonometry, probably about 55,000 of the S. A. T. C. men in all the colleges took that subject. Excluding the classes in trigonometry conducted for naval men only, there were 257 instructors, 63 per cent. of whom were on the regular mathematical staffs, 20 per cent. were from other departments, and 17 per cent. were temporary appointments. The replies indicated unanimously that the work in trigonometry was not as efficient as in their usual classes, nor in 75 per cent. of the institutions as extensive as usual. At all but two of the 29 schools, the chief reason assigned for the inferior work was absence for military duty, while poorer preparation was assigned as one of the reasons at half the schools and the influenza at several. At 18 of these 29 schools, surveying was taught to 3,664 S. A. T. C. men on an average of six hours per week of field work and three of indoor class work; the work was neither as effective nor as extensive as usual, due only partly to poorer preparation, but unanimously and emphatically attributed to cuts for military duty. On the average there were four men in a squad and 26 men per instructor; only 40 per cent. of the instructors were on the regular staffs. Navigation was taught to 2,170 men in sections of usually not over 25 men and with four recitations a week. Smaller numbers took courses in firing data, gunnery, ballistics, aerodynamics, statistics, as well as various subjects in mathematics. But the scheme partially failed because the lack of available experienced officers required that its execution be left usually to officers of very recent vintage, who were unable to understand why other young prospective officers needed the college courses, even though prescribed by the War Department, and, instead of regarding cuts from classes as a breach of military discipline, proceeded to remove men from their classes and assign them to all sorts of minor, menial, and clerical duties.

The ultimate object of exterior ballistics is to obtain data for range tables and the various ballistic corrections for practical use in directing the fire of the gun. When we entered the war we had no range tables for various types of guns we decided to adopt, especially for the anti-aircraft guns. The construction of the necessary new range tables involved not only the obtaining of a vast amount of experimental data, but also the elaboration of the theory of the differential equations which takes into account not only the resistance of the air but also its temperature and its decrease in density at higher altitudes, as well as corrections for the wind. Under the leadership of two of our well known mathematicians, Professors F. R. Moulton, and O. Veblen, now Majors in the Ordnance Department, two groups of mathematicians including Alexander, Bennett, Blichfeldt, Bliss, Buck, Dines, Gronwall, Hart, Haskins, Jackson, MacMillan, Milne, H. H. Mitchell, Ritt, Roever,

H. L. Smith, and Vandiver, have been engaged in this important work at Washington and Aberdeen, Md. For given initial conditions as to the gun, ammunition, elevation, and on the assumption of normal air density and no wind, the trajectory is now computed in about half a day, with a gain in accuracy. Some further simplification appears to result from the use of the adjoint system of differential equations. But it would be foolish for me to attempt to go into details since you are to have the pleasure of hearing this afternoon five ballistic experts who come to us direct from the two centers of ballistic work in America. Following my request for some information suitable for use in this address, I received from both Washington and Aberdeen huge bundles of blue prints showing hundreds of beautiful trajectories and other curves and many heavy mathematical manuscripts,—a sort of long range bombardment which it seemed the part of wisdom to dodge and trust to the direct fire of the newly arrived experts.

Believing that navigation should receive more attention in future in collegiate instruction, I shall give an outline of certain mathematical aspects of the subject.

By dead reckoning is meant the determination of the position of a ship by means of the measured distances and courses which it has sailed from a known position P. The true course C is the angle made by the ship's track with the north and south line. The method of plane sailing is employed when the distance D sailed is so short that we may neglect the curvature of the earth. Hence we have a plane right triangle with hypotenuse D, one angle C, the vertical leg being the difference of latitude expressed in nautical miles, and the horizontal leg being called the departure, as in surveying. Thus the legs are

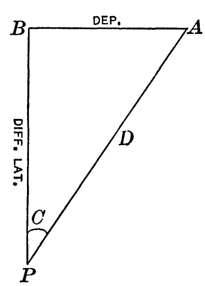

Fig. 1.

$$\text{Diff. Lat.} = D \cos C, \qquad \text{Dep.} = D \sin C,$$

and may be computed by logarithms or found as in surveying by inspecting a traverse table in which are entered the products of each number $D = 1, 2, \cdots, 600$ by $\cos C$ and by $\sin C$ for $C = 1^\circ, 2^\circ, \cdots, 89^\circ$. It remains to find the difference of longitude, i. e., the arc of the equator intercepted by the meridians through P and A, the point from which we sailed and the point arrived at. The east and west arc through A which is intercepted by those two meridians is the departure. Since these two arcs subtend equal angles at their centers, their ratio equals the ratio of the radii of their circles, which is immediately seen to be the secant of the latitude. Hence Dep. = (Diff. Long.)(cos Lat. A), from which we may find the difference of longitude by logarithms or by a traverse table.

In middle latitude sailing we take into account the curvature of the earth and assume that the ship's track is a rhumb or loxodromic line making the same angle C with all the meridians crossed. Divide the distance D into parts each so small that it can be regarded as the hypotenuse of a plane right triangle with an angle C. The sum of the vertical legs of these small triangles is seen to equal the difference of latitude, so that we again have Diff. Lat. = $D \cos C$. Since the meridians converge towards the north pole, the sum of the east and west legs of our small triangles has a value which exceeds the departure at A and is less than the departure at P and is assumed to equal the departure in middle (or mean) latitude, i. e., the east and west arc intercepted by the meridians through P and A on the parallel of latitude half way between the parallels of latitude at P and A. If these parallels are far apart or if either is near a pole, the assumption just made introduces too large an error. When the assumption is valid, we have Dep. in Middle Lat. = $D \sin C$. Hence we may proceed exactly as in plane sailing with departure replaced by departure in middle latitude.

Mercator's sailing involves no assumption restricting its accuracy and has the further advantage that the computations can be conveniently checked graphically on a chart which shows the ship's position at all times and hence its relation to possible danger points. The earth's surface is mapped on the interior of a rectangle in such a way that the meridians are represented by parallel straight lines, as also are the parallels of latitude. Since the rhumb line on which we sail crosses all the meridians at the same angle C, it is mapped as a straight line. The act of plotting as straight lines the earth's meridians which converge at the poles has caused an opening out of these meridians, i. e., a stretching of east and west lengths. But we desire that any small figure on the map shall be of the same shape as the corresponding figure on the earth, even though it be magnified. Hence there must be simultaneously a stretching of north and south lengths, the amount of stretching being the secant of the latitude. If L is the latitude of a point on the earth, the number m of nautical miles in the magnified latitude (called meridional parts) is given by a table, which is computed by use of the formula

$$m = r \log \tan (45^\circ + \tfrac{1}{2}L) \\ - r(e^2 \sin L + \tfrac{1}{3} e^4 \sin^3 L + \tfrac{1}{5} e^6 \sin^5 L + \cdots),$$

where r is the equatorial radius and e is the eccentricity of the ellipse whose rotation produces the earth's surface, while Naperian logarithms are employed. Taking for simplicity the earth to be a sphere, a small length $r\,dL$ on a meridian is represented on Mercator's map by $r \sec L\,dL$, whence the length on the map of the meridian from the equator to latitude L is

$$\int_0^L r \sec L\,dL = r \log \tan (45^\circ + \tfrac{1}{2}L).$$

By making use of the table of meridional parts we can readily construct to scale a rectangular Mercator's chart showing for example the parallels of latitude for 20°, 21°, $\cdots$, 30° North latitudes and the meridians for 70°, 71°, $\cdots$, 85° West longitudes; the entire rectangle is therefore divided into 10×15 small rectangles with equal bases, but varying heights which increase as we pass to higher latitudes. Such position charts are published by the U. S. Hydrographic Office. On a Mercator map, angles are the same as the represented angles on the earth, and difference of longitude is found by the scale at the bottom of the large rectangle. But as distances and differences of latitude appear magnified, the lines representing them are measured to the scale appropriate to their latitude, such varying scales being often given in the right and left hand margins directly opposite to the latitude.

For the computation by logarithms or a traverse table, we use the plane right triangle on a Mercator's map whose legs are the meridional difference of latitude and the difference of longitude and one angle is the course C, as well as the formula Diff. Lat. = Dist. $\times \cos C$ derived above from the curvilinear triangle on the earth. We make no use of the side "departure" in the last triangle, or of the hypotenuse of the former.

Great circle sailing is employed on very long voyages since the distance sailed is then a minimum, although it has the inconvenience that the course is changing continually. We first plot the track from port to port on a gnomonic projection chart (a projection on a tangent plane from the center of the earth), on which the great circle track is represented as a straight line. Then we transfer the route to a Mercator's map. In 1858, Sir George Airy proposed a method of representing approximately a great circle directly on a Mercator's map (Bowditch, page 82). Great circle distances and courses are found by spherical trigonometry, as in the later discussion of the astronomical triangle. An account of the literature on great circle sailing has been given by G. W. Littlehales.*

Owing to various inaccuracies in the data used in dead reckoning, the navigator must correct his estimated position by use of sights or observations of the sun or stars. We proceed to explain the method now in general use.

Suppose that a navigator measures with a sextant the sun's altitude (its angle of elevation above the horizon) and finds it to be 70°, so that the sun's zenith distance z is 20°. Then he is 20° or 1,200 nautical miles from the geographical position U of the sun, i. e., the point on the earth having the sun in its zenith. Hence the ship lies on a small circle whose spherical radius is 1,200 miles and spherical center is U. This circle of equal altitudes is in practice replaced by the tangent line, called a Sumner line of position, which is perpendicular to the bearing of the sun. It was discovered in 1837 by an American shipmaster, Capt. T. H. Sumner,† under the stress of saving his ship from imminent danger. Two special cases of the method had long been in constant use. The navigator took a sun sight just after sunrise and just before sunset to determine his longitude, the Sumner line then being perpendicular to the approximately East or West bearing of the sun. He took a noon sight to find his latitude, the Sumner line then being perpendicular to the North or South bearing of the sun.

In 1875 Admiral Marcq Saint Hilaire‡ of the French navy gave the following method to find the Sumner line. Given

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* The Development of Great Circle Sailing, U. S. Hydrographic Office, 1889, No. 90.

† A New and Accurate Method of Finding a Ship's Position at Sea, Boston, 1843; third ed., 1851.

‡ Calcul du point observé, *Revue Maritime et Coloniale*, vol. 46, 1875, p. 341, p. 714.

the estimated position A of the ship as found by dead reckoning and the geographical position U of the sun (or a star), we compute the great circle distance AU by one of the formulas below, and either compute the bearing (azimuth) of U from A or take it from a table of azimuths or from Weir's Azimuth Diagram. Then $h = 90° - AU$ is the computed altitude. Let h' be the sun's altitude observed with the sextant. On a Mercator's chart lay off from A the difference of the altitudes in miles towards U or in the opposite direction from U according as h is less than or greater than h'. Then the straight line through the point B thus determined and at right angles to the bearing is the Sumner line* containing the ship's true position.

The computation is made by use of formulas derived from the astronomical triangle MPZ, whose projection on the plane of the horizon is shown in Fig. 2, in which M represents the sun (or star), P the elevated pole, and Z the observer's zenith (point overhead). The declination d of the sun at the moment

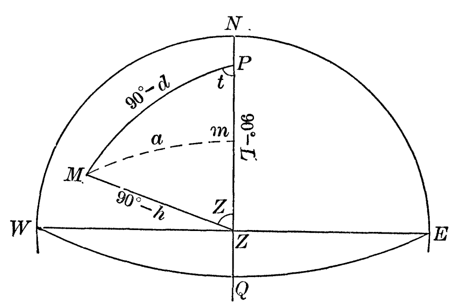

FIG. 2.

of observation is given by the Nautical Almanac; its hour angle t and the observer's latitude L are supposed known. A standard formula of spherical trigonometry expresses $\cos MZ$ in terms of the remaining two sides and their included angle t:

$$\cos(90° - h) = \cos(90° - L)\cos(90° - d) + \sin(90° - L)\sin(90° - d)\cos t.$$

Replacing $\cos t$ by $1 - 2\sin^2 \frac{1}{2}t$, we get

$$\sin h = \cos(L - d) - 2\cos L \cos d \sin^2 \tfrac{1}{2}t.$$

* Approximately. See the report below on Guyou's tables.

By finding the final product by logarithms, we readily get h. It is customary to use a formula obtained from the last by introducing versine x for $1 - \cos x$ and haversine x for $\frac{1}{2}$ vers $x = \sin^2 \frac{1}{2} x$. Since $h = 90° - z$, we get

$$1 - \operatorname{vers} z = 1 - \operatorname{vers}(L - d) - 2 \cos L \cos d \operatorname{hav} t,$$

whence, by cancellation and division by 2, we finally have*

$$\operatorname{hav} z = \operatorname{hav}(L - d) + \cos L \cos d \operatorname{hav} t.$$

Table 45 in Bowditch's American Practical Navigator gives the haversines and their logarithms of angles at intervals of 15 seconds of angle (or one second of time) up to 120° (or 8 hours), with an extension to 180°. Since we now have the three sides of our triangle MPZ, we may compute the azimuth angle Z by use† of

$$\cos^2 \tfrac{1}{2} Z = \cos s \cos (s - p) \sec L \sec h, \quad s = \tfrac{1}{2}(L + h + p),$$

where p is the polar distance $90° \pm d$.

In 1875, Lord Kelvin‡ stated that it ought to be the rule and not the exception to use Sumner's method for ordinary navigation at sea.

We may solve our astronomical triangle PZM by use of the spherical traverse table published by Commander F. Radler de Aquino§ of the Brazilian Navy. In Fig. 2, let the perpendicular a from M to PZ divide the latter into the parts $Pm = 90° - b$ and $Zm = 90° - B$. Use is made of six formulas given by Napier's rules. By means of

$$\sin d = \cos a \sin b, \qquad \cot t = \cot a \cos b,$$

* C. L. Poor, in his Simplified Navigation for Ships and Aircraft, 125 pp., 1918, N. Y., The Century Co., describes his machine to compute z by this formula. It is in effect a circular slide rule with several circular discs for finding the separate terms of the formula.

† Or by a formula for its haversine, Muir, Navigation, 1918, p. 444; Card, Navigation Notes, p. 9 (example, p. 90).

‡ Popular lectures and addresses by Sir Wm. Thomson, vol. 3, "Navigational Affairs," 1891 and 1894, Macmillan and Co.

§ The "Newest" Navigation. Altitude and Azimuth Tables, 2d ed., London, J. D. Potter, 1917. Aquino, O methodo de Marcq Saint Hilaire, Rio de Janeiro, 1902; Typos de calculo . . . , 1902. The table was reprinted and illustrated by examples (but without explanation of its construction) in Altitude, Azimuth and Line of Position, U. S. Hydrographic Office, 1917, No. 200.

he computed and tabulated the values of d and t corresponding to values of b for every degree and of a for every 30′ from 0° to 84° and for every 1° from 84° to 90°. Since

$$\sin h = \cos a \sin B, \qquad \cot Z = \cot a \cos B$$

are of the same form as the preceding equations, the values of h and Z for given a and B are already known. Hence the table has a double set of labels

$$\underline{B\backslash b \mid h\backslash d \mid Z\backslash t \mid}$$

at the top of the page for any given value of a. Finally,

$$\sin a = \cos d \sin t, \qquad \cot b = \cot d \cos t$$

show that we have automatically tabulated the values of a and b (marked by labels at the bottom of the page) which correspond to given values of d and t. The table has a column with the heading $C\backslash c$ showing $c = 90° - b$ or $C = 90° - B$; also a column showing the two angles at M. To do away with certain interpolations, we take an assumed latitude and longitude nearly the same as those given by dead reckoning, without changing the accuracy of the Sumner line.

Lord Kelvin* had previously published a smaller table of the same type.

F. Souillagouët's† final table of 108 pages is a traverse table for the right spherical triangle MmP of Fig. 2. It gives as entries $\phi = Pm$ and a for arguments t (angle at P) and hypotenuse $90° - d$, each at intervals of 30′ up to 90°. Having a and $mZ = PZ - \phi$, we again enter the table and read off the azimuth Z and altitude h. His first table of 254 pages serves to solve triangle PZM; let $\phi' = ZK$ be the perpendicular from Z to PM, and let $\phi = PK$. Then

$$\tan \phi = \cot L \cos P, \qquad \sin h = f \cos(90° - d - \phi),$$
$$f = \sin L/(\cos \phi) = \cos \phi'.$$

The table gives ϕ and $\log f$. We then get h.

* W. Thomson, Tables for facilitating of Sumner's method at sea, London, 1876. Cf. H. Jacoby's Navigation, 1917, pp. 126–133, 292–317.

† Tables du point auxiliaire pour trouver rapidement la hauteur et l'azimut estimés, suivies d'un recueil nouveau de tables nautiques . . . , new ed., Toulouse, 1900.

G. W. Littlehales* published a book of charts which serve to solve graphically the astronomical triangle PZM. Employ a stereographic projection of the celestial sphere on the plane of the observer's meridian (a projection from the pole of the meridian circle). By use of the latitude, $90° - PZ$, mark the observer's position Z on the bounding meridian. Locate the position M of the observed celestial body by means of its declination $90° - PM$ and its hour angle MPZ. In the triangle

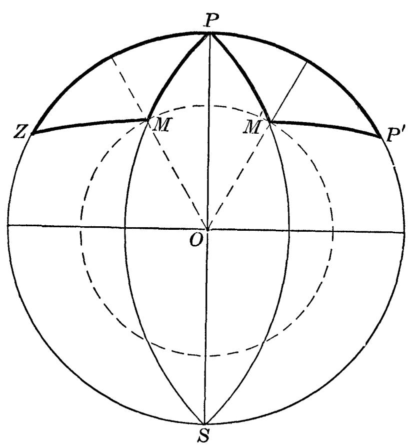

FIG. 3.

PMZ we have two sides and the included angle and desire the azimuth PZM and co-altitude ZM. Rotate the triangle about the center O of the projection with the side PZ kept in coincidence with the bounding meridian until Z is brought to the position of P, whence P is moved to a position P', and M to M'. The co-altitude now lies along a meridian PM' and the azimuth becomes the angle $M'PP'$ between two meridians, so that each can be measured by means of the graduations of the projection. To obviate the necessity for the actual rotation of the triangle, draw a series of equally spaced circles with the center O, numbered serially from O outward, and a series of equally spaced radial lines, marked by numbers indicating their angular distances in minutes of arc counted in clockwise direction from OS. After plotting M, note the num-

* Altitude, Azimuth and Geographical Position, comprising Graphical Tables . . . , Philadelphia, Lippincott, 1906, $25.

ber of the circle and the number of the radial which pass through M. To the number of the radial add the number of minutes in the co-latitude. The sum is the radial number of M', which is thus located in the circle just noted. We can now read off from the graduated arcs of the projection the desired altitude and azimuth. The projection was constructed for a sphere 12 feet in diameter and subdivided into 368 sections. The plate for a section is about a foot square and two of them are printed on the large page. There is a diagram which furnishes an index to the plates. For example, let the declination be 45° 54′ N, the hour angle 30° 43.5′, and the latitude 39° 16′ N. Plotting the declination and hour angle roughly on the index of plates with reference to the parallels and meridians (counted from the left-hand bounding meridian), we find that the position of the observed body falls on plate No. 63 approximately at the intersection of circle 17.2 with radial 8,400. The co-latitude is 3,044 minutes. Thus the approximate position of the rotated position M' is the intersection of circle 17.2 with radial 8,400 + 3,044 and hence falls on plate No. 258. Turn to plate No. 63 and plot the declination and hour angle carefully; we find that M is at the intersection of circle 495.6 with radial 8,411. Then M' on plate No. 258 is at the intersection of that circle with radial 8,411 + 3,044, whence we read off the altitude 66° 36′ and the azimuth N 63° 32′ W. The method applies at once to sailing on a great circle from Z to M, the initial course being angle PZM. In the problem to identify an observed star, we know its altitude and azimuth and hence point M'; we get M and hence its declination and hour angle.

E. Guyou* recently published extensive tables for the accurate simultaneous determination of altitude and azimuth. Underlying his method are geometrical facts of considerable interest. On the sphere let CC_1M be a circle of position with the center A and let Z be the position of the ship estimated by dead reckoning (Fig. 4). Let I be the intersection of the circle with the great circle AZ. On Mercator's chart, let cc_1m and zia be the curves which represent the circle CC_1 and the great circle ZIA (Fig. 5). The true line of position ih is the normal at i to the arc zi. But by Saint Hilaire's method we draw the tangent zj at z to the arc zi and take a perpendic-

* Nouvelles Tables de Navigation, Paris, 1911. Vol. 1, Réduction à l'Equateur, 33 + 370 pp. Vol. 2, Calcul de la Hauteur et de l'Azimut, 284 pp.

ular jh' to this tangent as the line of position. While this line passes very near to i, its direction is in error by an angle hih' equal to the angle between the tangents at z and i to the

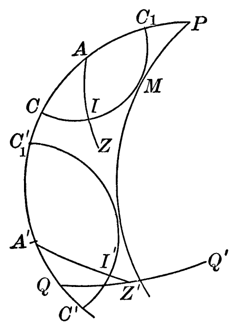

FIG. 4.

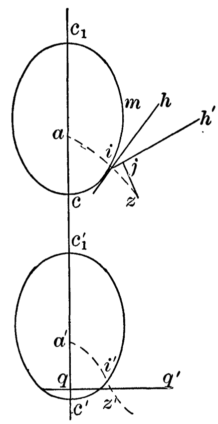

FIG. 5.

arc zi. This error increases with the latitude and practically disappears at the equator, i. e., in the case of circle $C'C_1'$ and point Z' of Fig. 4, since the great circle arc $Z'I'A'$ is represented on the chart (Fig. 5) by a curve $z'i'a'$ which has an inflexion at z' and hence coincides with its tangent for a considerable length. The last fact is the basis of Guyou's method to find a line of position which presents all the advantages of the line of Saint Hilaire and yet is free from the imperfections with which the latter line is in general affected. Starting with the "figure" (cc_1, z) composed of the curve cc_1 and the point z, slide it down to occupy the position $(c'c_1', z')$. This displaced figure represents on the sphere a figure composed of a circle $C'C_1'$ and point Z', for which Saint Hilaire's method is practically exact, since $z'i'a'$ is practically straight near z'. The method consists of two operations,—reduction to the equator and determination of the altitude and azimuth for the reduced figure at the equator, being accomplished by tables 1 and 2 respectively. First, let

$$H = 90^\circ - CA,\ D = 90^\circ - PA,\ P_e = ZPA,\ L = 90^\circ - PZ$$

be the true altitude, declination and hour angle of the observed body and latitude of the estimated position Z, and hence the "real data." Let the corresponding values for figure $(C'C_1', Z')$, with Z' on the equator, be

$$H' = 90° - A'C', \quad D' = QA', \quad P_e' = P_e = Z'PA', \quad L' = 0,$$

which are the "reduced data." By use of relations like

$$qc = \log \tan\left(\frac{\pi}{4} + \frac{QC}{2}\right),$$

it is proved in his article* (but not stated in his book) that

$$\cot\frac{H' + D'}{2} = t\cot\frac{H + D}{2}, \quad \cot\frac{H' - D'}{2} = \frac{1}{t}\cot\frac{H - \mathrm{D}}{2},$$

$$t = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right).$$

Table I gives the resulting values of $\frac{1}{2}(H' \pm D')$ as functions of $H \pm D$ for each L. By use of a right triangle, we get $\tan Z_e = \cot D' \cdot \sin P_e$, where Z_e is the azimuth. Table II gives the values of H and Z as functions of D'.

Some Further Books on Navigation.

If all the books on navigation were collected together they would sink a ship. The following books in English are not afraid† of a needed mathematical formula and appear to be

* "Nouvelle méthode pour déterminer les droites de hauteur et le point observé," *Revue maritime*, vol. 180, Feb., 1909, pp. 223–266. Also, *Rivista marittima*, Aug., 1909.

† The following avoid explicit formulas: H. Marshall, Nav. made easy, Milwaukee, 1877. W. C. Bergen, Practice of Nav. and Naut. Astr., 671 pp., 9th ed., 1902, North Shields. L. Young, Simple Elements of Nav., 248 pp., 2d ed., 1898, N. Y. A. C. Johnson, On Finding the Lat. and Long. in Cloudy Weather, 39 pp., 22d ed., 1900 (and 1903), London, Potter; and a Handbook for Star Double-Altitudes, 32 pp., 1898. H. Taylor, Modern Nav., 347 pp., 1904, San Francisco. C. E. MacArthur, On Nav. simplified, 121 pp., 1906, Rudder Pub. Co. C. L. Poor, Naut. Science in its Relation to Practical Nav., 329 pp., 1910, Putnam's Sons, $2. J. Pendlebury, Plain Everyday Nav., 44 pp., 1911, Yonkers, $2.50. W. J. Smith, The Self-instructor in Nav., 138 pp., 3d ed., 1912, Bellingham, Wash., $3. H. L. Thompson, Self-instruction in Nav., 80 pp., 1916, Portland, Me., $1.50. H. Jacoby, Nav., 350 pp., 2d ed., 1917, Macmillan, $2.25. G. L. Hosmer, Nav., 214 pp., 1918, Wiley, $1.25. S. T. S. Lecky, Wrinkles in Practical Nav., 18th ed., 1918, Van Nostrand, $10. F. S. Hastings, Modern Nav. by Sumner-St. Hilaire Methods, 84 pp., 1918, Appleton and Co. W. J. Henderson, Elements of Nav., new ed., 1918, Harper and Bros., $1.25. A. G. Mayor, Nav. Illustrated by Diagrams, 207 pp., 1918, Lippincott, $1.50. The last six books are good of their kind.

suitable as college texts: W. C. P. Muir, A Treatise on Navigation and Nautical Astronomy, 784 pages, 4th edition, 1918, U. S. Naval Institute, Annapolis, the text-book at the U. S. Naval Academy (contains 160 pages on the compass and the theory of its deviations). W. R. Martin, Navigation and Nautical Astronomy, 429 pages, 3d edition, 1899, Longmans, Green and Company. J. H. C. Coffin, Navigation and Nautical Astronomy, 221 pages, 7th edition, 1898, Van Nostrand. F. C. Stebbing, Navigation and Nautical Astronomy, 328 pages, 1896 and 1903, Macmillan and Company. J. Gill, Text-book on Navigation and Nautical Astronomy, 380 pages, 1898 and 1905, Longmans, Green and Company. W. Hall, Modern Navigation, 378 pages, 1904, Clive, London. V. J. English, Navigation for Yachtmen, 342 pages, 1896, H. Cox, London. Card's book was cited above.

The following three books are good ones of the Bowditch type: H. Raper, Practice of Navigation and Nautical Astronomy (1840), 934 pages, 19th edition, 1899, London, J. D. Potter; M. F. Maury, A new Theoretical and Practical Treatise on Navigation, 520 pages, 3d edition, 1864, Philadelphia; J. H. Colvin, Nautical Astronomy, 127 pages, 1901, London, Spon.

Among very clear, elementary books are those by W. T. (Earl of) Dunraven, Self-instruction in the Practice and Theory of Navigation, Macmillan, two volumes, 1900, three volumes, 1908, $8; D. Wilson-Barker and W. Allingham, Navigation, Practical and Theoretical, 154 pages, 1896, $1.50, London, Griffin and Company; J. Merrifield, Treatise on Navigation for Use of Students, 305 pages, 5th impression, 1900, Longmans, Green and Company (dead reckoning only); F. S. Hastings, Navigation, 109 pages, 1917, 75 cents, Appleton (no Sumner line); J. R. Walker, An Explanation of the Method of Obtaining the Position at Sea known as the New Navigation, 67 pages, 1901, Portsmouth (Sumner line only). Many problems and examination papers occur in A. P. W. Williamson's Text-book of Navigation and Nautical Astronomy, 418 pages, 2d edition, 1915, Portsmouth; and in H. B. Goodwin's Problems in Navigation, two volumes, 1893–6, London, Philip and Son. Wentworth's text omits the subject of Sumner lines.

J. B. Harbord's Glossary of Navigation, 512 pages, 3d edition, 1897, Portsmouth, is an excellent nautical dictionary and reference book.

Of French books, the two by J. B. Guilhaumon, Astronomie et Navigation suivies de la Compensation des Compas, 611 pages, 5th edition, 1912, and Eléments de Navigation et de Calcul nautique, two volumes, 1891, and G. Lecointe, La Navigation astronomique et la Navigation estimée, 392 pages, 1897, all published by Berger-Levrault, Paris, seem excellent. Especially recommended are E. Perret's Navigation, Instruments, Observations, Calculs, 1908, Paris, O. Doin et Fils, 5 francs (in Encyclopédie scientifique); and Cours de Navigation et de Compas, 1913, Ecole Navale, A. Challamel, Paris, 10 francs.

Many German texts are listed in Katalog der Büchersammlung der Nautischen Abteilung des Reichsmarineamts, 1905, Berlin, Mittler und Sohn.

On tables of altitudes and azimuths, see Encyclopédie des Sciences mathématiques, volume VII, part 1, 218–223.

MINIMUM LIST* OF BOOKS ON THE MATHEMATICAL THEORY OF BALLISTICS.

Articles in Encyclopédie des Sciences mathématiques, volume IV, part 6, pages 1–191.

P. R. Alger and others, Ground Work of Practical Navy Gunnery or Exterior Ballistics, 329 pages, 1915, Annapolis, U. S. Naval Inst., $6.

P. Charbonnier, Traité de Balistique, second edition, 592 pages, 1904, Paris, Beranger; Balistique extérieure rationnelle, two volumes, 492 and 406 pages, 1907, Paris, Doin; Balistique intérieure, 351 pages, 1900, Paris, Doin.

C. Cranz, Lehrbuch der Ballistik,† Leipzig, Teubner; volume I, Aeussere Ballistik, second edition, 1917, 529 pages; volume III, Experimentelle Ballistik, 339 pages, 1913; volume IV, Tafeln und Photographieen, 81 pages, 1910.

J. M. Ingalls, Artillery Circular M, Ballistic Tables, Revised by Ordnance Board, 233 pages, 1918, Government Printing Office, Washington.

F. Siacci, Balistique extérieure, 474 pages, 1892, Paris, Berger-Levrault.

E. Vallier, Balistique extérieure, second edition, 212 pages, Paris, Gauthier-Villars, 1913 (with improvements on Siacci's methods).

G. M. Wildrick, Gunnery for Heavy Artillery, Coast Artillery School, Ft. Monroe, 1918 (recent methods for effect of wind on trajectories).

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* Prepared by Professors Haskins and Gronwall, with the concurrence of Professor Bliss. As supplementary books they suggest A. Hamilton, Ballistics, Artillery School Press, Ft. Monroe, 1908–9; J. M. Ingalls, Interior Ballistics, third ed., 1912, Wiley and Sons, $3; Ingalls, Artillery Circular N, Problems in Exterior Ballistics, 250 pp., Government Printing Office, 1900 (essentially a revised form of his Handbook of Problems in Direct and Indirect Fire, 1890, Wiley & Sons); I. Didion's and N. Mayevski's texts, 1860 and 1872, of historical interest. Also Journal of the U. S. Artillery, Ft. Monroe, 1892–, and U. S. Naval Institute Proceedings, Annapolis, 1874–, each with many translations of articles in foreign artillery journals.

† English transl. by Greenhill just announced.

BOOKS ON THE THEORY OF AVIATION.*

* W. L. Cowley and H. Levy, Aeronautics in theory and experiment, Longmans, 1918.

A. Fage, The Aeroplane, London, Griffin and Company, 1917.

M. A. S. Riach, Airscrews, New York, Appleton, 1916.

* F. Lanchester, Aerodynamics, 1907; Aerodonetics, 1908, Van Nostrand, $6 each (classics though rather out of date).

Duchêne, The mechanics of the aeroplane, Longmans.

A. Klemin, Course in aerodynamics and airplane design, New York, Gardner, Moffat and Company, $5 (from *Aviation and Aeronautical Engineering*, 1916–7).

F. E. Bedell, Airplane characteristics, Ithaca, Taylor and Company, 1918.

In addition to these particularly recommended books, access should be had to the following:

* Technical Reports of the British Advisory Committee for Aeronautics, London, yearly (Greenhill's articles, of special interest to mathematicians, have been published separately under the titles* Dynamics of Mechanical Flight, 1912, Van Nostrand, and Report on Gyroscopic Theory).

* Annual Reports of the American Advisory Committee for Aeronautics, Washington (the articles by Hunsaker, Durand and E. B. Wilson are of special interest to mathematicians).

* G. H. Bryan, Stability in Aviation, 1911, Macmillan, $2.

S. Brodetsky, *Tôhoku Math. Journ.*, June, 1916; Aug., 1918.

* J. C. Hunsaker and others, Dynamical Stability of Aeroplanes, Smithsonian Miscellaneous Collections, 62, 1916, No. 5.

* N. Joukowski, Bases théoretiques de l'Aéronautique: Aérodynamique, 230 pp., 1916, Gauthier-Villars.

As regards the experimental work, recommended books are:

* G. Eiffel, Résistance de l'air, 1910, 5½ francs; Resistance de l'air et l'aviation, 1912, 12 francs (transl. by Hunsaker, Resistance of the Air, Houghton, Mifflin and Company, 1913); Nouvelles recherches . . . faites au laboratoire d'Auteuil, 1914, 50 francs.

J. C. Hunsaker, Stable Biplane Arrangements, *Engineering*, Jan., 1917.

* Hunsaker and others, Reports on Wind Tunnel Experiments in Aerodynamics, Smithsonian Inst., 1916, 30 cents.

Armand de Gramont, Essais d'Aérodynamique, Gauthier-Villars, 1911–4.

* A. W. Judge, Properties of Aerofoils and Aerodynamic bodies, 1917, Macmillan, $6.

R. Pannell, The wind channel, *Aeronautical Journ.*, July, 1918.

* F. Lanchester, Flying-machine from engineering standpoint, 1916, Van Nostrand, $3.

* S. P. Langley, Experiments . . . , Smithsonian Inst., 1891, 1908, 1911 (of historical interest only).

* L. Marchis and Riccardo-Brauzzi, in their three and two volume works each entitled Cours d'Aéronautique, treat the theory of balloons very fully.

I must omit the list of titles of over fifty papers of mathematical character which appeared during the past two years

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* Recommended by Professor H. Bateman of the Aeronautical Laboratory of Troop College of Technology, Pasadena, Cal. The starred books are those recommended independently by Professor E. B. Wilson, who remarked that A. F. Zahm's Aerial Navigation, Appleton and Company, 1911, gives the best popular historical account, while the best elementary account is by Painlevé and Borel, l'Aviation.

in the following journals: *Aviation, Aeronautics, Aeronautical Journal, Aerial Age, Proceedings of the Royal Society of London*, volume 91A, page 354, page 503, and *Aviation and Aeronautical Engineering* (which gave two lists of books on aviation, August 15, 1916, page 31, May 1, 1917, page 306).

A PARTIAL ISOMORPH OF TRIGONOMETRY.

BY PROFESSOR E. T. BELL.

1. It is well known that the only continuous solution, $\varphi(x)$, $\psi(x)$, of the system of functional equations

$$\varphi(a+b) = \varphi(a)\varphi(b) - \psi(a)\psi(b), \tag{1}$$

$$\psi(a+b) = \psi(a)\varphi(b) + \varphi(a)\psi(b), \tag{2}$$

$$\varphi^2(a) + \psi^2(a) = 1 \tag{3}$$

is $\varphi(x) \equiv \cos x$, $\psi(x) \equiv \sin x$. By suppressing the condition that φ, ψ, shall be continuous functions of a single variable, and one or two of (1), (2), (3), we get what may be called the partial isomorphs of trigonometry, whose interest, of course, will depend chiefly upon their interpretations. Several such are known and in use. While seeking arithmetical paraphrases for some of the more complicated results in elliptic and theta functions, I noticed incidentally another of these isomorphs in which (1), (2) only are retained. Apart from its usefulness in the theory of numbers (which is not considered here), this isomorph is of interest because it extends, in a sense, the concepts of evenness and oddness, or parity, relatively to functions of several variables.

2. We denote the sets of variables, $(x_1, x_2, \cdots, x_n)$, $(-x_1, -x_2, \cdots, -x_n)$ by ξ, $-\xi$ respectively, and define a function f to be even or odd in ξ according as it does not or does change sign when the signs of $x_1, x_2, \cdots, x_n$ are changed simultaneously: $f(\xi) = f(-\xi)$, or $f(\xi) = -f(-\xi)$, according as f is even or odd in ξ. These relations may be written $f((x_1, x_2, \cdots, x_n)) = \pm f((-x_1, -x_2, \cdots, -x_n))$, the inner ( ) being used to distinguish $f(\xi)$ from $f(x_1, x_2, \cdots, x_n)$, in which no property of evenness or oddness is implied.