2. $\sum_1^\infty \frac{a^n}{1\cdot 3\cdot 5\cdots(2n-1)} \cos(1\cdot 3\cdot 5\cdots 2n-1)\pi x$

$|a| > 1 + \frac{3}{2}\pi$ (Dini).

3. $\sum_1^\infty \frac{a^n}{1\cdot 5\cdot 9\cdots(4n+1)} \sin(1\cdot 5\cdot 9\cdots(4n+1))\pi x$

$a > 1 + \frac{3}{2}\pi$ (Dini).

4. If $\sum_1^\infty \frac{a_i}{10^i}$ denote any non-terminating decimal,

$$\sum_0^\infty \frac{a_i}{10^i} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (10^{3^i} x\pi).$$

5. $\sum_0^\infty \frac{1}{a^n} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (n!a^n\pi x)$, where $|a|$ is an integer > 1.

6. $x\sum_0^\infty a^n \sin b^n x\pi$, $|a|<1$, $|ab|>1+\frac{3}{2}\pi$, has a derivative for $x = 0$ but for no other value of x.

7. $\sum_0^\infty \frac{x^n}{n!} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (n!\pi x)$ has derivatives *between* -1 and $+1$ and no derivatives if $|x| > 1 + \frac{3}{2}\pi$.

Lerch gives a theorem* which shows that this last function has no derivatives for any *rational* points for which $|x| \geqq 1$. It is easy to show that it can have a finite derivative for no point $|x| > 1 + \frac{\pi}{2}$.

A HALF CENTURY OF FRENCH MATHEMATICS.

Les Sciences Mathématiques en France depuis un Demi-Siècle. Par Emile Picard. Paris, Gauthier-Villars, 1917. 24 pp.

In the first decades of the last century the home of the scientific spirit was in France. Paris was the capital of the Republic of exact truth. Interest in scientific discovery and creation was widespread among her people. The spirit of literature flourished alongside the spirit of exact researches

* Lerch, *Crelle's Journal*, vol. 103, p. 130 ("Ueber die Nichtdifferentiierbarkeit gewisser Funktionen").

and both found place in the same creative intellect. Out of this union of elements, too much separated in other countries, there grew up a tradition of literary excellence in scientific exposition which abides to the present and contributes in no small way to the comfort and delight which every one must feel in reading a French scientific book or memoir.

From the extreme precision and abstract character of mathematics it probably does not lend itself as readily to literary form as most other sciences; but, even here, the French have found a method of presentation which renders delightful both the books and memoirs of her masters. On account of their preëminence in this respect it is probably this which first comes to mind when one thinks of the last half century of French mathematics.

Yet the actual and detailed contributions of French investigators to the body of recent discoveries in mathematics looms large in quantity and bears the distinguishing mark of the highest quality. Indeed, leading mastery in the art of presentation can grow out of nothing other than a penetrating insight into the secrets of a subject.

Judged in the light of the excellence of French exposition that which is current in America is seen to fall well below the standard to which we should seek to attain. Among us there is a dearth of expository treatments, both of the sort by which one may get his introduction to a chapter of modern mathematics and of that by which he may obtain a well-rounded initial view of a comprehensive subject. Our memoirs are often printed in such condensed form that even practised mathematicians must waste in reading them time which might have been saved by the use of larger space in setting forth a more illuminating exposition. There is also a tendency towards the publication of matter which is not sufficiently well digested in thought. Results at the half-way stage towards the goal of an investigation are harder to understand than those of the ultimate reach which come of a more penetrating analysis. Too many memoirs appear profound because the writers have not discovered the simpler and more fundamental truths. The French are at present the leading masters in avoiding this defect and attaining the opposite virtue.

The production of work of great value requires extended continuity of spirit and effort. In recent French mathematics this truth is illustrated by the intimate relation of later discoveries to the earlier work of Fourier, Cauchy, and Galois. A profound study of nature was the source of the fruitful mathematical discoveries of Fourier, who found in physics the first origins of his great analytical theories and who developed analysis primarily for its usefulness to him as a physicist. His work, admirable for its clarity, contained the germ of important methods since employed in the fuller development of the theory of differential equations and of the expansion of arbitrary functions. At the opposite extreme was Galois with his investigation of the theory of algebraic equations and their groups. His researches were far removed from practical experience and even from contact with any theory of nature. He was engaged in problems belonging to the most abstract of pure mathematical disciplines. Between these two and reaching to both extremes was found the prodigious activity of Cauchy, whose researches extended to every domain of pure and applied mathematics. His greatest creation was that of the theory of functions of a complex variable.

During the past hundred years—and no less in its latter half than in its former—the theory of functions of a complex variable has been in great honor among the French. To an extent unusual in our generation we have here a vast subject dominated by the spirit of a single people. It is significant of the character of French thought that this theory of functions of a complex variable is perhaps the most elegant theory of like extent in the whole domain of mathematics.

The pamphlet under review is reprinted from a work published in 1916 with the title Un Demi-Siècle de Civilisation française (1870–1915) and thus covering the period between the present and the last great war fought by France. After an introduction of three pages, Picard gives here a brief analysis of the French mathematics of the period, presenting his matter under the following headings: analytic functions (pages 4–9); differential equations (pages 9–14); theory of numbers, algebra and geometry (pages 14–19); theory of functions of real variables and theory of assemblages (pages 19–23); some final remarks (pages 23–24).

During the last fifty years an important part of French mathematical effort has been devoted to the theory of functions both in its general aspects and as regards certain special classes of functions. Among those who have been interested in the general theory we may cite the names of Laguerre, Poincaré, Picard, Appell, Goursat, Painlevé, Hadamard, Borel. The demonstration of the Cauchy theorem concerning the zero value of an integral along a contour had formerly assumed the continuity of the derivative; Goursat has shown that this hypothesis is unnecessary. Picard has established the fact that in the neighborhood of an isolated essential singularity a function takes infinitely often every given value with the possible exception of at most *two* values; many authors have drawn various consequences from this remarkable theorem. We may also call attention to the developments, in series of polynomials, of Appell and Painlevé, to the functions with lacunary spaces of Poincaré and Goursat, and to the more recent developments of Montel. Concerning the fundamental character of a power series on the circle of convergence are to be found Hadamard's classic and far-reaching investigations. Laguerre's study of entire functions, Borel's treatment of summable series, and Poincaré's work on multiform functions are other matters to which attention should be directed.

In our period the most important work dealing with special classes of functions has been that of Poincaré on automorphic functions. The fuchsian functions afford a means of uniform parametric representation of any algebraic curve, "certainly one of the most profound results obtained in analysis in the last fifty years."

In recent years Borel has made a penetrating study and comparison of the notion of *analyticity* in the sense of Weierstrass and of that of *monogeneity* in the sense of Cauchy, and has reached the conclusion that the latter is the essential one. The detailed investigations on which this conclusion is based are set forth in Borel's Leçons sur les Fonctions monogènes uniformes, 1917. The essential results, however, were announced at the Cambridge Congress in 1912.

In the hands of Picard the method of successive approximation has been a useful tool in the study of differential equations. Poincaré has introduced the general notion of asymptotic representation for the solutions of difference equations and of differential equations in the neighborhood of an irregular singular point. Poincaré has shown how linear algebraic differential equations may be integrated by means of thetafuchsian series. Painlevé has done fundamental work in the classification of non-linear differential equations with fixed critical points. Poincaré has devoted several memoirs to the study of curves defined by differential equations and has made particularly notable contributions to the theory of the differential equations of physics. The work of Darboux on differential geometry is everywhere recognized as of leading importance. The same author has also created a new method of integrating partial differential equations; Goursat has devoted several papers to elucidating the questions suggested by this method.

The methods created by Hermite have given to the theory of numbers a new and broader horizon. The work of Jordan on the theory of forms has realized important progress in the general theory of algebraic forms of higher degree. Poincaré has left his mark on the theory of quadratic forms. Jordan has made a profound study of the epoch-making ideas of Galois. Halphen has extended the theory of twisted algebraic curves. Cartan has contributed to our knowledge of the structure of groups and to the determination of simple groups.

A considerable number of memoirs, of unequal value, have been devoted to the theory of assemblages. Picard believes that some of these are without interest to mathematics and characterizes their theory as a sort of *metamathematics* from which paradoxes are not absent. Of the important work in the theory of assemblages one may select for mention that of Borel on measure, that of Lebesgue on integration, and that of Baire on the classification of functions.

Sometimes mathematical researches are presented with an excessive formalism and symbolism, incapable of leading to a new fact and of being utilized in other researches than in those in which they were created. But when these last conditions are not realized one can think that there is no real scientific progress. In this respect it seems that French mathematics has wisely restrained itself within proper bounds and has not created a chapter of science which affords merely an exercise in logic. Whenever our science has a tendency to become too formal let us turn again to the work of the great creators, such as Fourier, Cauchy and Poincaré, and learn from them anew how to create other chapters in that body of mathematics which is not a strange and mysterious science, but an essential part of the structure of natural philosophy.

R. D. Carmichael.