This process may evidently be continued. We may then state the following

Theorem: The $r$th polar of $B$ with respect to $C_n$ is $C_{n-r}$.

II.

Again let there be three distinct points $A$, $B$, and $C$ on the same straight line $l$, and through the point $C$ let the line $l_1$ be drawn perpendicular to $l$. Let lines $l_2$ and $l_3$ be drawn through $A$ and $B$ respectively, and let $l_2$ and $l_3$ intersect on $l_1$. Let $l_2$ make an angle $\alpha$ with $l$, and $l_3$ make an angle $\beta$ with $l$, and let a line $l_4$ be drawn through $B$, making an angle $n\beta$ with $l$. Let $l_2$ and $l_4$ intersect in $D$. Then just as in section I, the equation representing the locus of $D$ is

$$(7)\qquad k\left[x^n - \binom{n}{2}x^{n-2}y^2 + \cdots\right] = (x - c)\left[\binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3}y^2 + \cdots\right],$$

where $k = (a - c)/a$ and $a = AC$, and $c = AB$.

It is then evident that the theorem in section I holds for the curve represented by equation(7).

OHIO STATE UNIVERSITY.

---

# ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY DR. MARY F. CURTIS

(Read before the American Mathematical Society, April 27, 1918.)

GIVEN the twisted cubic

$$(1)\qquad x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0;$$

to show that the condition that it is a helix is precisely the condition that it is algebraically rectifiable.

If (1) is a helix, then $T/R$, the ratio of curvature to torsion, is constant. Denoting differentiation with respect to $t$ by primes, we have

$$\begin{array}{llll} x': & a & 2bt & 3ct^2, \\ x'': & 0 & 2b & 6ct, \\ x''': & 0 & 0 & 6c, \end{array}$$

$$(x' \mid x') = a^2 + 4b^2t^2 + 9c^2t^4, \qquad (x'' \mid x'') = 4(b^2 + 9c^2t^2),$$

$$(x' \mid x'') = 2t(2b^2 + 9c^2t^2), \qquad |\, x'x''x''' \,| = 12abc,$$

$$(x'x'' \mid x'x'') = (x' \mid x')(x'' \mid x'') - (x' \mid x'')^2$$
$$= 4(a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4).$$

$$\frac{T}{R} = -\left(\frac{x'x'' \mid x'x''}{x' \mid x'}\right)^{3/2} \frac{1}{|x'x''x'''|}.$$

Since $|x'x''x'''|$ is constant, $T/R$ is constant when and only when $(x'x'' \mid x'x'')/(x' \mid x')$ is constant. We thus have

$$4(a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4) \equiv \rho(a^2 + 4b^2t^2 + 9c^2t^4);$$

hence $\rho = 4b^2$ and $9a^2c^2 - 4b^4 = 0$. Conversely, for all values of $a,b,c$, $abc \neq 0$, for which

(2) $$9a^2c^2 - 4b^4 = 0,$$

$T/R$ is constant—in particular, is equal to $\mp 1$, according as $2b^2 = \pm 3ac$—and the cubic (1) is a helix.

If we had fixed our attention on another characteristic property of a helix, namely, that the tangent makes with a fixed direction a constant angle, we should have again derived the condition (2). The fixed direction—that of the axis of the cylinder on which the helix lies—is $(1/\sqrt{2}, 0, \pm 1/\sqrt{2})$ and the helix cuts the rulings of the cylinder under an angle of 45°.

That (2) is a necessary and sufficient condition that $s$, the arc of (1), is an algebraic function of $t$ and hence that (1) is algebraically rectifiable follows from the fact that the integral

(3) $$s = \int_{t_0}^{t} \sqrt{a^2 + 4b^2t^2 + 9c^2t^4}\,dt$$

is algebraic when and only when (2) holds. Hence the theorem: The twisted cubic (1) is algebraically rectifiable when and only when it is a helix.