It is honorable alike to the Royal Academy of the Lincei and to the colleagues of Cremona and younger mathematicians, that they unite to preserve in worthy form the works of a justly celebrated scientist and leader. Eighteen already have shared the not inconsiderable labor of thorough editing, and their corrections and explanatory notes, appended to each volume, form a valuable aid to the reader. The highest tribute that can be paid to the memory of a scientist is the labor that makes his work more useful to the next generation.

HENRY S. WHITE.

BLICHFELDT'S COLLINEATION GROUPS.

Finite Collineation Groups. By H. F. BLICHFELDT. Chicago, University of Chicago Press, 1917. 12mo. 12 + 194 pp.

THIS little volume forms a notable contribution to the series of mathematical texts by American authors that have appeared in recent years. Coming from the pen of an author who has an unusual mastery of his subject, it is moreover almost unique in its field, the promised text by Wiman for the Teubner series (as far as the reviewer is aware) not having appeared. Certain parts of the subject, particularly the theorems depending on the invariance of a Hermitian form and the theory of group characteristics, may be found in the second edition of Burnside's Theory of Groups, which appeared in 1911. A considerable part also of the material in the present treatise may be found in Part II of Finite Groups, by Miller, Blichfeldt, and Dickson, which was written by the same author.

On the other hand there is much in the present volume that cannot be found elsewhere except in scattered journal articles, and some of the results at the close of Chapter IV seem to be entirely new. The author's own share in the development of the subject is a very notable one, the theorems in Chapter IV concerning the linear groups in n variables being almost entirely his own. In addition the complete determination of the groups in three and four variables was first made by him, the earlier work along this line being reproduced in Chapters V and VII in a somewhat revised form. There are in addition chapters on the "Elementary Properties of Linear Groups," "Groups of Operators and Substitution Groups," "The Linear Groups in Two Variables," "The Theory of Group Characteristics," and "The History and Applications of Linear Groups."

As the author remarks in the closing chapter, the theory of linear groups may be said to have originated with Klein, who was mainly interested in their application to the solution of algebraic equations. At about the same time, however, they were employed by several writers, notably Schwarz, Fuchs, and Jordan, in the study of linear differential equations having algebraic integrals. The theory received in this way such an impetus that it has now grown to considerable proportions.

The problem of the determination of the finite collineation groups has been attacked in two ways. One may ask either "What groups may be represented in a given number of variables?" or "In how many ways may a given abstract group be represented as a linear group?" Concerning the first of these questions there is the theorem of Jordan that the order of a finite linear group in n variables is of the form λf, where f is the order of an abelian self-conjugate subgroup, and where λ is inferior to a fixed number that depends only upon n. The theorems of Chapter IV enable the author to obtain a definite limit for λ; in fact, he was the first to obtain such a limit, although other (much higher) limits have been obtained by Bieberbach and Frobenius. The exact values that λ may have are known only for $n = 2, 3, 4$.

The theory of characteristics, which is due mainly to Frobenius, throws considerable light on the second question. One of the theorems obtained by this means is to the effect that if a given abstract group of order g be represented as a "regular" group of permutations on g variables, and then by means of a change of variables the group be represented by means of a series of "transitive" component groups, every possible representation of the given abstract group as a transitive linear group will occur in this series and the number of times it occurs is equal to the number of variables in which it is represented. Moreover, the number of such representations that are "non-equivalent" is equal to the total number of sets of conjugate operators of the given abstract group.

The author includes these theorems in his chapter on the theory of group characteristics. It seems unfortunate, however, that he should have used the terms "equivalent or non-equivalent groups" instead of "equivalent or non-equivalent representations of a group." This point will be discussed more in detail later.

There seems no doubt that the author was desirous of developing the subject of linear groups proper as far as space would permit, and for this reason has omitted any extended discussion of the geometrical properties of the individual groups, such as their invariants. The methods used in the determination of the groups in three and four variables are as a rule analytical, although geometrical arguments have been employed in a few places.

In spite of the fact that the treatment was evidently not intended to be exhaustive, the reviewer was somewhat disappointed not to find anything concerning the groups that are known to exist in more than four variables. Perhaps the most interesting groups of this sort are the three systems in p^m, $(p^m - 1)/2$, and $(p^m + 1)/2$ variables that are isomorphic with the "abelian linear group" on $2m$ indices, the coefficients of which are residues, modulo p. For $m = 1, 2$ these groups have been considered by Klein and several others in their relation to elliptic and hyperelliptic functions. The ternary groups of order 216, 168, 60 and the quaternary groups of order 11520, 25920, 168 each belong to one of these systems. There are in addition groups in six, seven, and eight variables, reference to which might have been included among the historical notes in the closing chapter.

In taking up the consideration of the various chapters somewhat more in detail, special mention should be made of the fundamental theorem, due to several authors, that every finite linear group in n variables has an invariant definite Hermitian form. This theorem, which is included among those given in the first chapter, is used to establish the result, due originally to Maschke and later extended by Loewy, that every "reducible" group is "intransitive." Thus, for example, every collineation group in three-dimensional space that has an invariant plane must also leave fixed a point not in that plane. This result is used in turn to show that every linear group in which the transformations are commutative may be written in canonical form. It might perhaps have been well to direct the reader's attention to the fact that this is not necessarily the case for collineations.

The second chapter contains an introduction to the theories of abstract groups and permutation groups, introduced mainly for the benefit of readers who may have had no previous acquaintance with the subject. Under the former heading we find for example Sylow's theorem and some discussion of abelian groups and groups having for order a power of a prime. With regard to permutation groups there is included something concerning the representation of a given abstract group as a "regular" group and some theorems on the alternating and symmetric groups.

For the determination of the linear groups in two variables the author has chosen on account of its historical and geometrical interest the process employed by Klein, whereby the possible groups are shown to be isomorphic with the groups of rotations of the regular solids. An outline is also given of Jordan's process for finding these groups by means of a diophantine equation. Another somewhat similar equation which would answer the same purpose may be obtained from Theorem 4 of the chapter on group characteristics. It follows from this that the order g of any transitive collineation group in two variables must satisfy the equation

$$g = 4 + g\sum_i f_i(n_i - 2)/n_i,$$

where the summation is to be taken over the orders $n_1, n_2, \cdots$ of the cyclic subgroups in the different conjugate sets, and each $f_i = 1, 2$.

The fourth chapter is the one that is likely to attract the widest attention. One of the most interesting theorems that it contains is to the effect that the order of a primitive group in n variables cannot be divisible by a prime greater than $(n-1)(2n+1)$, a result first published by the author in the *Transactions* in 1903. Whether for large values of n there exist groups whose orders are divisible by primes nearly as large as this seems rather doubtful; in fact, as far as the reviewer is aware, no primitive group is known to exist whose order is divisible by a prime greater than $2n+1$. It is a real achievement, however, to have been able to obtain any limit at all.

In the analysis employed in the proof a certain equation involving only roots of unity is shown to exist connecting the characteristics of certain transformations. By use of Kronecker's theorem concerning the irreducibility of the general cyclotomic equation the author ingeniously transforms this equation into a congruence, from which it follows that the product of any two transformations of order p, a prime exceeding the limit given above, must be either of order p or identity. For special values of n he is able to reduce this limit. For $n = 3, 4$ it is found that no primes exceeding 7, 13 respectively can divide the order of a primitive group.

The author's statement of Kronecker's theorem (§133), or rather the consequence that he deduces from it, is not quite accurate. For example, the rational equation of lowest degree that is satisfied by θ, a primitive 15th root of unity, is

$$\theta^8 - \theta^7 + \theta^5 - \theta^4 + \theta^3 - \theta + 1 = 0,$$

but the seven roots of unity appearing in this sum cannot be separated into sets of the sort that he describes. This does not form a serious defect in his argument, however, since it does follow from Kronecker's theorem that every sum of roots of unity that is equal to zero may be thrown into the form (4), page 85, by the addition and subtraction of roots that cancel each other by pairs. The second paragraph under 7°, §133, is subject to a similar modification.

There are also theorems that limit the conditions under which a primitive simple group can contain transformations having for order a power of a prime and commutative transformations of different orders. In Theorem 7, page 93, condition (B) is unnecessary, since there do not exist more than m different roots of unity whose mth powers are all equal to the same root.

In the latter part of the chapter an interesting type of analysis first employed by Bieberbach and Frobenius is used by the author to show that no primitive group can contain a transformation whose multipliers when represented graphically on the unit circle all lie on an arc that does not extend more than 60° on either side of some one of them. This theorem is found useful in the chapters on the groups in three and four variables. It is also used in the section that immediately follows to establish an upper limit for the order of any abelian subgroup that can be contained in a primitive group in n variables.

Finally by use of the various theorems of the chapter a limit is obtained for the order of a primitive group in n variables, a result that has been referred to above. This limit is lower (as far as the reviewer is aware) than any that have been published previously. The two theorems moreover that were referred to in the preceding paragraph seem also to be new, although the first of these bears a close relation to one obtained by Frobenius.

In the succeeding chapter these results are applied to the determination of the primitive groups in three variables. The first accurate solution of this problem was obtained by the author of this text, the previous attempts by Jordan and Valentiner being only partially successful. These groups are found to be the "Hessian" group of order 216, so called by Jordan on account of its relation to the inflexional points of a pencil of cubic curves, two of its subgroups of orders 72 and 36, and three simple groups of order 60, 168, 360. The last two were first found in the order named by Klein and Valentiner.

Although a number of expositions of the theory of group characteristics have been given, notably by Frobenius, Burnside, Schur, and Dickson, the author does not follow closely any of them. In making essential use of the invariance of a Hermitian form his treatment probably resembles Burnside's more than any of the others.

Some of the theorems here give additional information concerning the possible primitive groups in n variables. By means of one of them it follows that if any group contains transformations of orders p and q, two distinct primes each greater than $n + 1$, it must also contain transformations of order pq. As far as the reviewer is aware, no primitive group of this sort is known to exist. Another useful relation is that for a transitive group the sum of the products of the characteristics and their conjugate imaginaries (for each of the transformations) is equal to the order of the group. An application of this theorem to the determination of the binary groups has been mentioned above.

Some of the theorems concerning the representation of a given abstract group as a linear group have already been referred to. To illustrate the sense in which the author uses the terms "equivalent or non-equivalent groups," we consider the linear group of order 27 generated by $x' = y$, $y' = z$, $z' = x$, and $x' = x$, $y' = \omega y$, $z' = \omega^2 z$, where $\omega^2 + \omega + 1 = 0$. We may put the transformations of this group into (1, 1) correspondence with those obtained by replacing ω by ω^2, which latter form therefore a group simply isomorphic with the first. According to the author's terminology (Theorem 5, page 128) these two groups are "non-equivalent," doubtless for the reason that it is impossible by a change of variables to make the *corresponding* transformations simultaneously identical. On the other hand they contain exactly the same transformations in the form in which they are written, though arranged in a different order.

This seems to the reviewer an unusual use of these terms. As the author apparently does not mean the definition on page 64 to be taken in this sense, it seems not unlikely that the reader may place a wrong interpretation on some of the theorems in this chapter. That this is the meaning that must be attached to the word "equivalent" may be seen from a careful examination of the proof of Theorem 5.

An interesting application of the theory of characteristics is made at the close of the chapter to prove the well known theorem that every group of order p^aq^b is composite.

In Chapter VII the author applies the theorems of Chapters IV and VI to the determination of the primitive groups in four variables. The exposition of the subject in this treatise has been somewhat modified from the original one, which appeared in the *Mathematische Annalen* in 1905, and which was an achievement of no mean order. If any primitive group is not itself simple (as a collineation group), then somewhere in its chain of factor groups there will occur either a primitive simple group or else a group that is not primitive, thus affording a basis for a classification.

It follows at once from geometrical considerations and by use of the knowledge concerning the groups in two and three variables that no primitive group in four variables can contain transformations with two pairs of equal multipliers of higher order than 5 or transformations with three equal multipliers (homologies) of higher order than 3. In a separate discussion it is shown that transformations of the former type having the order 5 may also be ruled out. The simple group of order 25920 is found to be the only primitive group that contains homologies of order 3.

The discussion at the bottom of page 145 may not be entirely clear to a reader. The author remarks that the two binary groups of order $60 \cdot 2$ must be "equivalent," since the two generating transformations of order 5 have the same multipliers in the two cases, the term being used doubtless in the same sense as in the preceding chapter. It does not seem evident without further consideration that the quaternary group might not be (2, 1) isomorphic with either of the two binary groups, in which case Theorem 5 of Chapter VI would not give the desired result. This is apparently the theorem that the author is using here, although it is not explicitly referred to.

The proof of Theorem 3 would have been much simplified if after establishing the existence of a subgroup of order 648 the author had used the fact that any two homologies of order 3 that are not commutative must generate a group of order 24 in which all four homologies are conjugate and which contains an invariant reflection. Thus a homology must be conjugate with any that are not commutative with it and hence in a primitive group with all others. Also any homology not in the subgroup of order 648 must be commutative with one of the nine reflections that this subgroup contains, from which it follows that there cannot be more than 40 homologies altogether. By a proper choice of coordinates one of the nine reflections commutative with "D" may be taken as "F."

In view of Maschke's theorem (page 23) and the fact that there is no larger collineation group in three variables containing a G_{216}, the argument on page 151 beginning, "We now write down, etc.," appears superfluous.

Having eliminated from consideration certain types of transformations the author is able to show that the possible Sylow subgroups that may occur in a primitive simple group are of a comparatively limited number of types. A number of the theorems of Chapters IV and VI are found useful. One device that is employed consists of the introduction of line coordinates and the application of some of these theorems to the resulting group in six variables. In the reviewer's opinion it would have been desirable to have given some explanation of the method of excluding transformations of order 9. The final conclusion is that the only primitive simple groups that exist are of order 60 (two types), 168, 360, 2520, 25920.

The only groups in this list that can be contained self-conjugately in larger groups are those that are isomorphic with the alternating groups on five or six letters. Each of these is contained by a group of twice its order isomorphic with the corresponding symmetric group. These in turn are not contained self-conjugately by any larger groups.

The primitive groups containing invariant intransitive subgroups are all found to leave invariant a quadric surface. Some of these can be enlarged by the addition of transformations that interchange the two sets of rulings. Finally it is shown that the only primitive groups that contain imprimitive invariant subgroups are the one of order 11520 and certain of its subgroups.

The final chapter on the history and applications of linear groups has been referred to above. The author gives an interesting sketch of the methods used in their application to the solution of algebraic equations and describes also their relation to linear differential equations having algebraic integrals.

The book appears in a neat, attractive form and the proofreading has been carefully done, only a few errata being found. Those that were noticed, together with some minor comments, are as follows:

P. 19 (next to the last line). Replace $\bar{x}_{n-1}$ by $\bar{x}_n$.
P. 20 (8th line). Replace one y_{n-1} by $\bar{y}_{n-1}$.
P. 20 (19th line). Replace f'_g by f.
P. 65 (next to last line). The value of Y should be $Y = x_1\bar{x}_2 + x_2x_1$.
P. 72. The numbers given here are the numbers of *rotations* rather than the numbers of the *axes*. There are only 10 different axes of period 3 and 6 of period 5.
P. 79 (7th line). Replace Y_2'' by Y_1''.
P. 90 (3d line from bottom). Replace $p_a \gtreqless kp$ by $p^a \geqq kp$.
P. 100. It is perhaps worth while to remark that the sets of intransitivity "of highest index" referred to at the middle of the page may not coincide with the "ultimate" sets of intransitivity referred to in the preceding discussion.
P. 110. The author apparently intended that T' should be of period 4. For this we may take $\alpha = 1, \beta\gamma = -1$.
P. 112 (6th line). Replace $2^3 \cdot 3^2 \cdot \phi$ by $2^3 \cdot 3^2$.
P. 113 (last line). A factor 1/7 has been omitted from the first expression for h.
P. 126 (middle). The expression in the second parenthesis should be $Y_2 + a_2Y_3$.
P. 154. The orders of Q for the groups (g), (j), (k) should be respectively 10ϕ, 75ϕ, 150ϕ.

In view of the unusual conditions prevailing at the present time the book will undoubtedly not attract the attention it deserves. It ought to serve however to stimulate interest in the subject at least in this country. There are few, if any, theories that are possessed of greater elegance or that offer a more direct challenge to the mathematical investigator.

As a text for class-room use it will be found very suitable except perhaps for those whose interests center mainly in the geometrical aspects of the subject. Some of the analysis in Chapter IV may possibly be found rather difficult by immature students, but by suitable omissions no trouble would be experienced. Not only the author but the publishers as well are to be congratulated on their part in the production of the book.

HOWARD H. MITCHELL.

SHORTER NOTICES.

Elliptic Integrals. By HARRIS HANCOCK. New York, Wiley, 1917. 104 pp.

THIS volume is number 18 of the well-known series of Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. It was prepared in response to a request from the editors for a work of about one hundred octavo pages on elliptic integrals which should "relate almost entirely to the three well-known elliptic integrals, with tables and examples showing practical applications." The monograph is confined to the Legendre-Jacobi theory and the discussion is limited almost entirely to the elliptic integrals of the first and second kinds.

After a short introduction (pages 5–8), mostly historical, there follows in Chapter I (pages 9–23) an elementary discussion of the three kinds of elliptic integrals and the Legendrian transformations. The Jacobi elliptic functions are treated in Chapter II (pages 24–40). Chapter III (pages 41–64) is devoted to elliptic integrals of the first kind and Chapter IV (pages 65–87) to numerical computation of the elliptic integrals of the first and second kinds and to Landen's transformations. Several miscellaneous examples and problems are given in Chapter V (pages 88–91). In the sixth and last chapter (pages 92–101) are three five-place tables as follows: Table I, the complete integrals of the first and second kinds, page 93; Table II, elliptic integrals of the first kind, pages 94–97; Table III, elliptic integrals of the second kind, pages 99–101.