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INTEGRALS OF LEBESGUE.

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THE purpose of this paper is to present in detail the contents of an introductory lecture given by the writer at the symposium of the American Mathematical Society held in Chicago in April of this year on the subject of Lebesgue integrals. It would be impossible to have selected a subject for that occasion more characteristic of present mathematical tendencies. Volterra has pointed out, in the introductory chapter of his Leçons sur les Fonctions des Lignes, the rapid development which is taking place in our notions of infinite processes, examples of which are the definite integral limit, the solution of integral equations, and the transition from functions of a finite number of variables to functions of lines. In the field of integration the classical integral of Riemann, perfected by Darboux, was such a convenient and perfect instrument that it impressed itself for a long time upon the mathematical public as being something unique and final. The advent of the integrals of Stieltjes and Lebesgue has shaken the complacency of mathematicians in this respect, and, with the theory of linear integral equations, has given the signal for a reexamination and extension of many of the types of processes which Volterra calls passing from the finite to the infinite.

It should be noted that the Lebesgue integral is only one of the evidences of this restlessness in the particular domain of the integration theory. Other new definitions of an integral have been devised by Stieltjes, W. H. Young, Pierpont, Hellinger, Radon, Fréchet, E. H. Moore, and others. The definitions of Lebesgue, Young, and Pierpont, and those of Stieltjes and Hellinger, form two rather well defined and distinct types, while that of Radon is a generalization of the integrals of both Lebesgue and Stieltjes. The efforts of Fréchet and Moore have been directed toward definitions valid on more general ranges than sets of points of a line or higher spaces, and which include the others for special cases of these ranges. Lebesgue and Hahn, with the help of somewhat complicated transformations, have shown that the integrals of Stieltjes and Hellinger are expressible as Lebesgue integrals. It seems unfortunate that these reductions should have been interpreted by some as an attempt to establish the Lebesgue integral in the unique position so long occupied by the integral of Riemann. Van Vleck has in fact remarked* that a Lebesgue integral is expressible as one of Stieltjes by a transformation much simpler than that used by Lebesgue for the opposite purpose, and the Stieltjes integral so obtained is readily expressible in terms of a Riemann integral, as is shown in § 6 below. Furthermore the Stieltjes integral seems distinctly better suited than that of Lebesgue to certain types of questions, as is well indicated by the original "problem of moments" of Stieltjes,† or by a generalization of it which Riesz has made in his remarkable discussion of a problem analogous to the determination of a function whose Fourier constants are given.‡

The conclusion then seems to be that one should reserve judgment, for the present at least, as to the final form or forms which the integration theory is to take. It is probable that the outcome may be a general theory of the type of those of Fréchet and Moore, having not one but a number of special instances with forms more adaptable to problems of various special types. However this may be, there can be no question as to the wide influence which the work of Borel, Lebesgue, and their followers is having upon the mathematical thought of the present time, and no question as to the notable advances

* "Haskins's momental theorem and its connection with Stieltjes's problem of moments," *Transactions of the American Mathematical Society*, vol. 18 (1917), p. 327.

† "Recherches sur les fractions continues," *Annales de la Faculté des Sciences de Toulouse*, vol. 8 (1894).

‡ Riesz, "Sur certaines systèmes singuliers d'équations intégrales," *Annaels Scientifiques de l'Ecole Normale Supérieure*, vol. 28 (1911), p. 33.

which have been made in the many domains of real function theory to which the Lebesgue form of integral is especially adapted. Not all of these can be recounted here, but it is hoped that the selection made will indicate the tendencies of the theory, and make the approach to its further results an easy one.

The theory of Lebesgue integrals has had an able expounder in the person of de la Vallée Poussin, who has published systematic treatments of portions of it at some five different times and places, indicated in the list of references at the end of this paper. It is characteristic of the formative state of the subject at the present time that each of his presentations has signalized new theorems, new points of view, or improved methods of proof. The account given in his recent lectures at the Collège de France, designated by the number VII in the reference list at the end of this paper, is especially complete and satisfying, though somewhat more sophisticated than those of his earlier disquisitions.

The latest device of de la Vallée Poussin in approaching the theory of measure is to develop first of all the theory of the measures of denumerable sums of intervals and of closed sets of points, and then to define in terms of them the measures of more general sets.* A closed and bounded set is always the portion of an interval remaining after the extraction of the interiors of a denumerable sum of non-overlapping intervals. I have been interested here to attempt to return again to the more direct methods of Borel and Lebesgue,† which found the theory of measure entirely upon the measurability of denumerable sets of intervals. With the aid of the improved methods of proof devised by de la Vallée Poussin it is possible to approach the subject in a way which seems to me to be especially concise and clear, and possible at the same time to establish without added complication the foundations of the theory of the positive additive functions of a point set, of which the measure function is a special case.

The notion of a function $g(e)$ real, single-valued, and additive on a class of point sets e, was first emphasized by Lebesgue‡ in his search for something which might be called the indefinite integral of a function of several variables, and the theory of such functions of a point set was subsequently much

* VII, Chapitre II.
† I, p. 46; II, p. 238; III, p. 106.
‡ IV, p. 361.

perfected by de la Vallée Poussin* and Radon.† A function $g(e)$ is said to be continuous if its value approaches zero with the diameter of e, and it is absolutely continuous if its value approaches zero with the measure of e. Every function which is absolutely continuous is clearly continuous, and it seems a waste of effort therefore to develop their common properties separately, as has usually been done. It turns out that it is as easy to proceed directly to the theory of functions which are only continuous, as it is to found the theory of continuous functions upon the absolutely continuous case. A number of the properties of these functions which de la Vallée Poussin establishes independently in order to prove the formula (33) of § 8 are immediate consequences of this formula when it is proved directly.

1. *Definition and Existence of Lebesgue Integrals.*

The relation between the integral of Lebesgue and the classical integral of Riemann may be seen somewhat intuitively by considering first a function $f(x)$ which is continuous and has only a finite number of maxima and minima on an interval ab. Let l_k $(k = 0, 1, \cdots, n)$ be a set of values with the properties

$$(1)\quad l_0 < \mu, \quad M < l_n, \quad 0 < l_k - l_{k-1} \leqq \epsilon \quad (k = 1, 2, \cdots, n),$$

where μ and M are the lower and upper bounds, respectively, of the values of $f(x)$ on ab. If the points l_{k-1}, l_k are marked on the y-axis, as in the figure, a corresponding set of points e_k is determined on the x-axis consisting of all the abscissas x for which the values of $f(x)$ lie between l_{k-1} and l_k. Let η_k be a value arbitrarily selected on the interval $l_{k-1}l_k$, and let $m(e_k)$ denote the sum of the lengths of the x-intervals in the point set e_k. Then

$$(2)\qquad S = \sum_{k=1}^{n} \eta_k m(e_k) = \sum_{i=1}^{p} f(\xi_i)\Delta x_i,$$

where the totality of intervals Δx_i $(i = 1, 2, \cdots, p)$ is that defined by the sets e_k $(k = 1, 2, \cdots, n)$, and where ξ_i is chosen in each interval Δx_i of e_k so that $f(\xi_i) = \eta_k$. The maximum δ of the lengths of the intervals Δx_i will approach zero with ϵ, and it follows that

* VI, and VII, Chapitre IV. † VIII, p. 1299.

$$\lim_{\epsilon=0}\sum_{k=1}^{n}\eta_k m(e_k) = \lim_{\delta=0}\sum_{i=1}^{p} f(\xi_i)\Delta x_i = \int_a^b f(x)dx.$$

The only value of this argument for the purposes of the present paper is to show how the usual definite integral limit may be obtained in a special case by a partition on the y-axis, as well as by the usual subdivision of the x-interval ab. A difficulty at once presents itself when it is attempted to extend this method of procedure to functions not necessarily continuous or having a finite number of maxima and minima. For such less special functions the set of points e_k will not in general consist of intervals, and hence will not have a well-

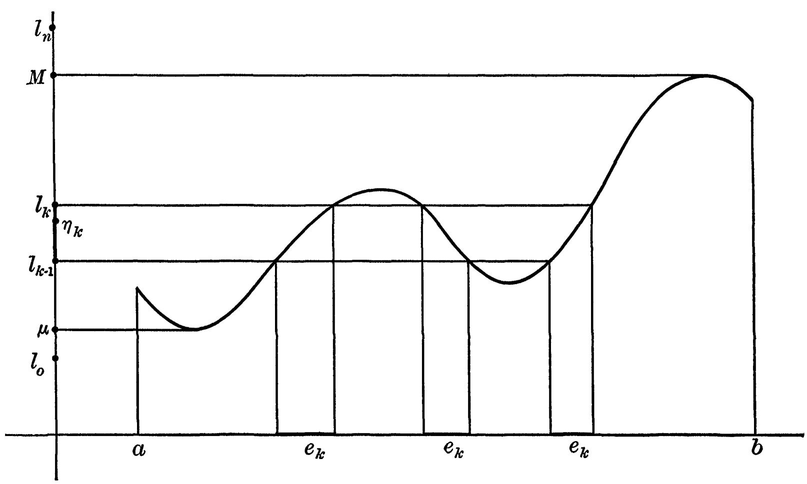

FIG. 1.

defined measure, unless some adequate definition of the measure of a point set has been previously set down.

The notions of the measure of a point set and measurable functions, which lie at the root of the Lebesgue theory of integration, will be examined in detail in later sections. For the present suppose that a definition of measure has been given which may not apply to all point sets, but which does apply to some at least, and let such sets be called measurable sets. Further suppose that the measure $m(e)$ of a measurable set e is always positive or zero, and that when $e_1, e_2, \cdots, e_n$ are measurable and distinct, the set $e_1 + e_2 + \cdots + e_n$ consisting of all their points taken together is also measurable, and

$$m(e_1 + e_2 + \cdots + e_n) = m(e_1) + m(e_2) + \cdots + m(e_n).$$

Let $f(x)$ be well defined and bounded at the points of a set e, and denote by $e[A \leqq f(x) < B]$ the points of e where $A \leqq f(x) < B$. Then f is said to be measurable on e if the set $e[A \leqq f(x) < B]$ is measurable for every pair of constants A, B.

DEFINITION OF A LEBESGUE INTEGRAL.* *Let $f(x)$ be a real single valued function which is measurable and has bounds μ, M on a point set e. Select a set of values l_k $(k = 0, 1, \cdots, n)$ with the properties* (1), *and let η_k be a value anywhere on the interval $l_{k-1}l_k$. Denote the set of points $e[l_{k-1} \leqq f(x) < l_k]$* by *$e_k$. Then by definition*

$$\int_e f(x)dx = \lim_{\epsilon=0} \sum_{k=1}^{n} \eta_k m(e_k). \tag{3}$$

To prove that the integral limit always exists under the circumstances described in the definition, consider first two sums S, S' like the one in the second member of (3), and suppose that the ladder of l-values for S' includes all of those of S. For simplicity of proof let the interval $l_{k-1}l_k$ of S be divided by only two points l', l'' in forming S', so that e_k is decomposed into three parts $e_k = e' + e'' + e'''$. Then the portions of S and S' corresponding to the interval $l_{k-1}l_k$ are

$$S: \quad \eta_k m(e_k) = \eta_k m(e') + \eta_k m(e'') + \eta_k m(e'''),$$

$$S': \qquad \eta' m(e') + \eta'' m(e'') + \eta''' m(e'''),$$

and they differ by less than $\epsilon\, e_k$ since

$$|\eta_k - \eta'| < \epsilon, \quad |\eta_k - \eta''| < \epsilon, \quad |\eta_k - \eta'''| < \epsilon.$$

This result would be the same if there were more or less than two new values l', l''. It follows then that

$$|S - S'| < \sum_k \epsilon m(e_k) = \epsilon m(e).$$

The constant ϵ of the definition may be called a norm for the sum S. All sums of this type lie between $(\mu - \epsilon)m(e)$ and $(M + \epsilon)m(e)$. Hence out of an arbitrarily selected sequence of sums with norms approaching zero, a sub-sequence $\{S_n\}$

* VII, p. 39; V, vol. 1, 3d edition, p. 257; III, p. 112.

may be chosen which has a limit I, and the corresponding sequence of norms $\{\epsilon_n\}$ will still have limit zero.

For every sum S with norm ϵ it follows then that

$$\begin{aligned} | S - I | &\leqq | S - S' | + | S' - S_n | + | S_n - I | \\ &< \epsilon m(e) + \epsilon_n m(e) + | S_n - I |, \end{aligned}$$

where S' is formed by using all the l-values of both S and S_n. But by taking ϵ sufficiently small and n sufficiently large, each term on the right can be made less than $\delta/3$, from which follows

THEOREM 1. *The Lebesgue integral limit* (3) *always exists for a function $f(x)$ which is measurable and bounded on the set e.*

§ 2. *Preliminary Properties of Point Sets.**

All of the point sets e to be considered are supposed to lie on a finite interval ab. It has been found convenient to define the sum $e_1 + e_2$ of two sets e_1, e_2 to be the totality of their points, the difference $e_1 - e_2$ to be the set of points which are in e_1 but not in e_2, and the product e_1e_2 to be the totality of points which e_1 and e_2 have in common. Addition and multiplication are readily seen to be commutative and associative, and to satisfy the relations

$$(e_1 + e_2)e_3 = e_1e_3 + e_2e_3, \quad (e_1 - e_2)e_3 = e_1e_3 - e_2e_3.$$

But one must be careful about subtraction. The sets

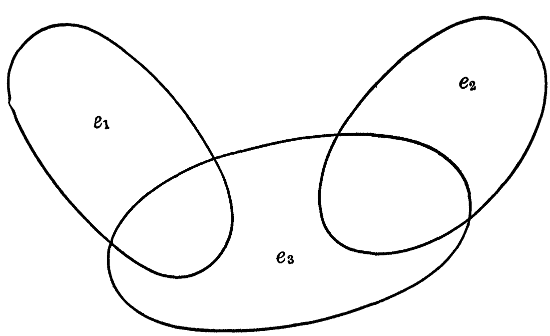

FIG. 2.

$(e_1 + e_2) - e_3$ and $e_1 + (e_2 - e_3)$ are not necessarily the same, as a glance at the accompanying figure for plane sets will show.

* VII, pp. 5 18; V, vol. 1, 3d edition, p. 59.

The complement Ce of a set e is the totality of points of the interval ab which are not in e. The difference and product of two sets e_1, e_2 are expressible in terms of addition and complements. For

$$(4) \qquad C(e_1 - e_2) = Ce_1 + e_2, \quad Ce_1e_2 = Ce_1 + Ce_2.$$

Hence theoretically the only operations which need to be considered are addition and the taking of complements. The others are, however, frequently found to be of great convenience.

The complete limit c of a sequence of sets $\{e_n\}$ is defined to be the totality of points each of which occurs in an infinity of the point sets e_n. It is representable by the formula

$$(5) \qquad c = (e_1 + e_2 + \cdots)(e_2 + e_3 + \cdots)\cdots.$$

The restricted limit r is the set of points for each of which there is a place in the sequence $\{e_n\}$ beyond which the point occurs in every e_n. A formula for r is

$$(6) \qquad r = (e_1e_2\cdots) + (e_2e_3\cdots) + \cdots.$$

The set r is always a part of the set c, and when the two are identical the sequence e_n is said to have a unique limit

$$(7) \qquad \lim e_n = c = r.$$

A particular case of the limits c and r is illustrated graphically for plane sets in Fig. 3. If all the sets e_n of odd index are identical with the vertical rectangle in Fig. 3, and all of even index identical with the horizontal rectangle, the complete limit is the totality of points of both, and the restricted limit is the square common to the two rectangles. On the other hand, if the sets of odd index shrink successively as indicated by the dotted lines, and similarly for those of even index, the complete and restricted limits are both identical with the corner square, and the sequence $\{e_n\}$ has this square as its unique limit.

There are two special cases for which the limit of a sequence $\{e_n\}$ is easily seen to exist. These are when each element e_n is contained in the following, and when each e_n is contained in the preceding. The limits are then

$$\lim e_n = e_1 + e_2 + \cdots, \quad \lim e_n = e_1e_2\cdots,$$

respectively. It follows that for every sequence $\{e_n\}$ whatsoever
$e_1 + e_2 + \cdots = \lim\,(e_1 + \cdots + e_n)$, $e_1 e_2 \cdots = \lim\,(e_1 e_2 \cdots e_n)$,
and furthermore from formulas (5) and (6)

$$(8)\qquad c = \lim\,(e_n + e_{n+1}\cdots), \quad r = \lim\,(e_n e_{n+1}\cdots).$$

LEMMA 1. If every point of a closed set e on the interval ab is interior to one of a set of intervals I, then there exists a finite number of the intervals I with the same property.

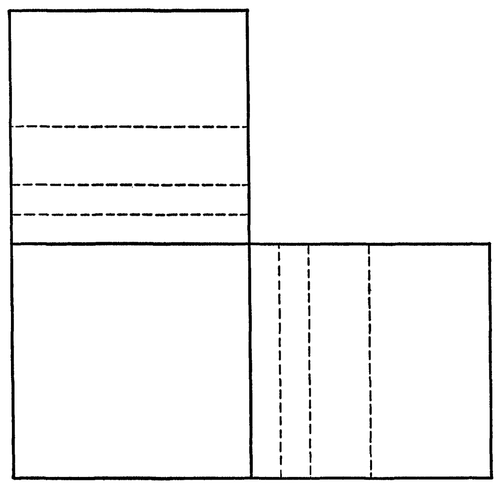

FIG. 3.

A closed set is one which contains all of its limit points. Every point ξ of ab is either in e, and therefore interior to an interval of I, or else is interior to an interval containing no point of e, since e is closed. When this remark is applied first at $\xi = a$, it is seen that there exists an interval ax such that all the points of e on ax are surely interior to a finite number of the intervals I. Let ξ next be the least upper bound of the values x defining such intervals. The same remark shows that if $\xi = b$ the lemma is true, while $\xi < b$ is impossible.

Consider now a function $P(x)$ which is continuous and monotonically increasing on the interval ab. If α is an interval with the end points $x_1 < x_2$, the notation $p(\alpha)$ will be used for the expression

$$(9)\qquad p(\alpha) = P(x_2) - P(x_1).$$

LEMMA 2. If the intervals α_k $(k = 1, 2, \cdots)$ are non-overlapping, and their points all among those of a second set β_k $(k = 1, 2, \cdots)$, then

$$\Sigma p(\alpha_k) \leqq \Sigma p(\beta_k).$$

The first series is surely convergent since the terms are all positive and the sum of the first m of them always less than $P(b) - P(a)$. If the second is finite then, since $P(x)$ is continuous, each β_k can be enlarged at each end to form an interval β_k' such that

$$p(\beta_k') < p(\beta_k) + \frac{\epsilon}{2^k}, \quad \Sigma p(\beta_k') < \Sigma p(\beta_k) + \epsilon.$$

The intervals $\alpha_1, \alpha_2, \cdots, \alpha_m$ form a closed set of points each interior to one of the intervals β'. Hence, by Lemma 1, these intervals are all enclosed in a finite set of intervals $\beta_1', \beta_2', \cdots, \beta_n'$, and it follows that

$$\sum_{i=1}^{m} p(\alpha_i) \leqq \sum_{k=1}^{n} p(\beta_k') < \sum_{k=1}^{\infty} p(\beta_k) + \epsilon.$$

Since this is true for every m and ϵ the inequality of the lemma must hold.

Point sets consisting of a denumerable set of intervals play an essential rôle in the definition of measure, as will be seen in the following pages. Let A and B always denote such sets, composed of the intervals α_k and β_k $(k = 1, 2, \cdots)$, respectively, the range of k being either finite or denumerably infinite.

LEMMA 3. A set of the type A is always expressible as a sum of a denumerable set of non-overlapping intervals.

For if the sum of the first n intervals $\alpha_1, \alpha_2, \cdots, \alpha_n$ of A is represented by a_n, it follows readily that

$$A = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots.$$

Furthermore each term in parenthesis can be expressed as a sum of a finite number of intervals not overlapping each other or the similar sums which precede.

The notation $p(A)$ will be used for the expression

$$p(A) = \sum_{k} p(\alpha_k),$$

where the intervals α_k are supposed to be non-overlapping. The number $p(A)$ so defined is uniquely determined by the set A. For Lemma 2 shows that

$$\Sigma p(\alpha_k) \leqq \Sigma p(\beta_k), \quad \Sigma p(\beta_k) \leqq \Sigma p(\alpha_k),$$

when the sums $\Sigma\alpha_k$ and $\Sigma\beta_k$ both consist of non-overlapping intervals and both define the same set of points A. Hence the value $p(A)$ is the same whatever set of non-overlapping intervals representing A is used to compute it.

THEOREM 2. *The totality of sets of the type A is closed under multiplication and addition, even for sums of a denumerable infinity of elements. The function $p(A)$ has the properties*

$$p(A) + p(B) = p(A + B) + p(AB), \tag{10}$$

$$p(A_1 + A_2 + \cdots) \leqq p(A_1) + p(A_2) + \cdots. \tag{11}$$

It is evident that the sum of a denumerable infinity of sets of the type A is a similar set, and the product AB also, since it is the sum of the product sets a_nb_n, where a_n and b_n are the sums of the first n intervals of A and B respectively. Furthermore the definition (9) of the function p for intervals shows that

$$p(a_n) + p(b_n) = p(a_n + b_n) + p(a_nb_n),$$

where the intervals of A, as well as those of B, may be supposed non-overlapping. But as n approaches infinity the terms of this equation approach as limits the corresponding terms of the equation (10), since $A + B$, for example, is the sum of the finite number of non-overlapping intervals constituting $a_n + b_n$, plus others which must be added to form $a_{n+1} + b_{n+1}$, and so on. The final conclusion of the theorem follows since every interval in the expression for $A_1 + A_2 + \cdots$, as a sum of non-overlapping intervals, occurs as a part or the whole of an interval in at least one of the sums A_k.

COROLLARY. *If A and B are entirely on the interval ab and include between them all of the points of ab, so that $A + B$ is the interval ab, then*

$$p(A) + p(B) = P(b) - P(a) + p(AB). \tag{12}$$

This follows from the property (10) above and the fact that $p(ab)$ is by definition equal to the first two terms on the right.

§ 3. *The Measure Function $m(e)$ and Similar Functions.*

In the preceding section a monotonically increasing continuous function $P(x)$ gave rise to a function $p(e)$ which was well defined for all point sets consisting of a finite or denumerably infinite system of intervals. When $P(x) = x$ the value of $p(e)$ for one of these sets is called the measure $m(e)$ of e. The purpose of the present section is the extension of the definition of $p(e)$, or $m(e)$, to a larger class of point sets than those of the special type A.

An arbitrarily selected point set e can be enclosed in infinitely many ways in a set of intervals of the type A on ab. The greatest lower bound of the values $p(A)$ for such sets is called the *exterior value of p on e*,* or in the special case $P(x) = x$ the *exterior measure of e*,† and may be denoted by $\bar{p}(e)$. It follows immediately from this definition that

$$
(13)\qquad \begin{gathered} \bar{p}(e) \geqq 0, \quad \bar{p}(e_1) \geqq \bar{p}(e_2) \text{ when } e_1 \text{ contains } e_2, \\ \bar{p}(e_1 + e_2 + \cdots) \leqq \bar{p}(e_1) + \bar{p}(e_2) + \cdots. \end{gathered}
$$

The last of these properties is valid for a finite or denumerably infinite sum, and is a consequence of the second conclusion of Theorem 2.

The exterior value $\bar{p}(e)$ could equally well be defined as the greatest lower bound of the values $p(A)$ on sets A restricted to contain the points of e as interior points. For when the intervals of A are enlarged slightly, as in the proof of Lemma 2, so that $p(A)$ is only slightly increased, the result is a set containing e in its interior.

The value $\bar{p}(e)$ might be taken as the measure of the set e, in the special case $P(x) = x$, if it surely possessed the property of additivity which was essential in the existence proof of § 1, and which is moreover characteristic of all notions of measure. The requirement that the measure of the whole shall be the sum of the measures of its parts is a fundamental one. It turns out that the class of point sets possessing the special additive property of the following definition is a class in which as a whole the function $\bar{p}(e)$ is additive.

DEFINITION OF A MEASURABLE SET.‡ *If the exterior values $\bar{p}(e)$ and $\bar{p}(Ce)$ satisfy the relation*

$$
(14)\qquad \bar{p}(e) + \bar{p}(Ce) = P(b) - P(a),
$$

* VII, p. 101.

† VII, p. 22; V, vol. 1, 3d edition, p. 61; III, p. 104.

‡ VII, pp. 22, 101; V, vol. 1, 3d edition, p. 62; III, p. 106.

then $p(e)$ is said to be well defined on e and its value is by definition

$$p(e) = \bar{p}(e) = P(b) - P(a) - \bar{p}(Ce).$$

When this happens for the special case $P(x) = x$ the set e is said to be MEASURABLE.

If a set of the type A encloses Ce then all points of the set $ab - A$ are interior to e. Lebesgue accordingly defines the interior measure of e in the case $P(x) = x$ to be

$$\underline{p}(e) = P(b) - P(a) - \bar{p}(Ce),$$

and calls the set e measurable if $\bar{p}(e) = \underline{p}(e)$. The definition of the preceding paragraph is clearly equivalent to this.

LEMMA 4. A necessary and sufficient condition that $p(e)$ be well defined on a set e is that there exist sets A, B enclosing e and Ce such that $p(AB) < \epsilon$.

The condition is clearly necessary. For suppose the relation (14) of the definition satisfied, and let A, B be chosen enclosing e, Ce, respectively, so that the first members of (12) and (14) differ by less than ϵ. Then a comparison of the second members shows that $p(AB) < \epsilon$. Conversely, suppose A and B can be chosen enclosing e and Ce for every ϵ so that $p(AB) < \epsilon$. Then the equation (12) shows that

$$\bar{p}(e) + \bar{p}(Ce) \leqq P(b) - P(a),$$

while the third of the relations (13) shows that

$$\bar{p}(e) + \bar{p}(Ce) \geqq \bar{p}(ab) = P(b) - P(a).$$

It is important to note, as a consequence of this lemma, that the value $p(A)$ of § 2 is precisely the value which would result from the definition last given. For from Lemma 2 and the fact that A encloses itself, it follows that $\bar{p}(A)$ is the $p(A)$ of § 2. Furthermore the portions B of the interval ab which are left after extracting the intervals $\alpha_1, \alpha_2, \cdots, \alpha_n$ enclose CA, and

$$p(AB) = \sum_{k=n+1}^{\infty} p(\alpha_k) < \epsilon$$

provided that n is sufficiently large, so that A is a set satisfying the requirements of Lemma 4.

The criterion of the lemma just proved is not always convenient of application. Another more useful one is that of the following theorem.

THEOREM 3. *Let A be a denumerable set of intervals enclosing e, and let e' be the part of A not in e, so that*

$$e + e' = A.$$

Then a necessary and sufficient condition that $p(e)$ be well defined is that for every $\epsilon > 0$ there exists a set A enclosing e with $\bar{p}(e') < \epsilon$. Under these circumstances the values $p(e)$, $p(A)$ satisfy the relation

$$p(e) \leqq p(A) < p(e) + \epsilon.$$

The condition of the theorem is necessary. For if A and B have been determined as in Lemma 4 so that $p(AB) < \epsilon$, then e' is in AB and will necessarily have $\bar{p}(e') < \epsilon$. To prove the sufficiency, suppose that a set A exists for which $\bar{p}(e') < \epsilon$. Then e' is enclosable in a set B_1 with $p(B_1) < \epsilon$. The rest of Ce exterior to A is enclosed in the intervals B_2 remaining in ab after $\alpha_1, \alpha_2, \cdots, \alpha_n$ have been extracted, and $p(AB_2) < \epsilon$ if n is sufficiently large. Hence Ce is enclosed in $B = B_1 + B_2$, and by (13)

$$p(AB) \leqq p(AB_1) + p(AB_2) < 2\epsilon.$$

A class $\mathcal{E}$ of point sets is said to be closed if it contains the complement of every one of its elements, and also the sum of every sequence $\{e_n\}$ whose elements are in $\mathcal{E}$. From the results of § 2 it follows at once that a closed class $\mathcal{E}$ is also closed under the operations of subtraction, multiplication of a finite or denumerably infinite set of elements, and taking complete or restricted limits of sequences.

A function $p(e)$ is said to be continuous if for every ϵ a δ can be found such that $p(e) < \epsilon$ for all sets e of diameter less than δ, and it is absolutely continuous if in this definition the word diameter is replaced by the word measure. It is said to be additive in $\mathcal{E}$ if

$$p(e_1 + e_2 + \cdots) = p(e_1) + p(e_2) + \cdots$$

for every finite or denumerably infinite set $\{e_n\}$ whose elements are distinct and in $\mathcal{E}$.

THEOREM 4. *The totality $\mathcal{E}$ of point sets e for which $p(e)$ is well defined according to the definition given above contains all intervals on ab and is closed. The function $p(e)$ itself is non-negative, continuous, and additive on $\mathcal{E}$. Furthermore for every set e of $\mathcal{E}$*

and every constant $\epsilon > 0$ *there exists a sum* A *of intervals enclosing* e *such that* $p(A - e) < \epsilon$.

It has been proved above that $p(e)$ is well defined on all denumerable sets of intervals A, and therefore also on every sub-interval of ab. Furthermore every element e_n of a sequence of sets in $\mathcal{E}$ is enclosable in a set A_n such that

$$e_n + e_n' = A_n, \quad \bar{p}(e_n') < \frac{\epsilon}{2^n}. \tag{15}$$

Hence

$$(e_1 + e_2 + \cdots) + (e_1' + e_2' + \cdots) = (A_1 + A_2 + \cdots),$$

and from (13) and (15)

$$\bar{p}(e_1' + e_2' + \cdots) < \epsilon,$$

which proves that $p(e)$ is well defined on the sum of the elements e_n.

The continuity of $p(e)$ follows from the fact that $P(x)$ is continuous, and from the second of the relations (13), which says that $p(e) \leqq p(\alpha)$ whenever e is contained in the interval α.

To prove the additive property, consider first only two distinct sets e_1 and e_2 of $\mathcal{E}$. According to Theorem 3, they can be enclosed in sets A_1, A_2 in such a way that

$$e_1 + e_1' = A_1, \quad e_2 + e_2' = A_2,$$

$$\bar{p}(e_1') < \epsilon, \qquad p(e_2') < \epsilon.$$

Then $p(e_1)$, $p(e_2)$, $p(e_1 + e_2)$ differ from $p(A_1)$, $p(A_2)$, $p(A_1 + A_2)$ by less than ϵ, ϵ, 2ϵ respectively. Furthermore, since e_1 and e_2 have no points in common,

$$p(A_1 A_2) = p(e_1'(e_2 + e_2') + e_1 e_2') < 2\epsilon,$$

$$p(A_1) + p(A_2) - p(A_1 + A_2) = p(A_1 A_2) < 2\epsilon. \tag{16}$$

It follows that

$$p(e_1) + p(e_2) - p(e_1 + e_2) = 0, \tag{17}$$

since the first member of this equation differs from the corresponding member of (16) by less than 4ϵ, and therefore from zero by less than 6ϵ, and ϵ is arbitrary.

The equation (17) implies the additive property for $p(e)$ for every finite number of sets e. When the sequence $\{e_n\}$ contains an infinity of elements, the series

$$p(e_1) + p(e_2) + \cdots$$

surely converges since

$$p(e_1) + \cdots + p(e_n) = p(e_1 + \cdots + e_n) \leqq p(ab).$$

From (17) also

$$p(e_1 + e_2 + \cdots) = p(e_1 + \cdots + e_n) + p(e_{n+1} + e_{n+2} + \cdots),$$

and with the help of (13)

$$p(e_{n+1} + e_{n+2} + \cdots) \leqq p(e_{n+1}) + p(e_{n+2}) + \cdots < \epsilon$$

if n is sufficiently large, which proves the desired property.

The last statement of the theorem follows at once from Theorem 3.

COROLLARY. *If e_2 is contained in e_1 and both are in $\mathcal{E}$, then*

$$p(e_1 - e_2) = p(e_1) - p(e_2).$$

If e is the limit of a sequence $\{e_n\}$ in $\mathcal{E}$, then

$$p(e) = \lim p(e_n).$$

The first conclusion is a consequence of the equation

$$e_1 = (e_1 - e_2) + e_2$$

and the additive property of $p(e)$. The second follows readily from the additive property when each e_n contains all the preceding, since then

$$e = e_1 + (e_2 - e_1) + (e_3 - e_2) + \cdots.$$

It is provable in a similar way when each e_n is contained in the preceding. For under that hypothesis

$$e = e_1 e_2 \cdots,$$

$$(18) \qquad Ce = Ce_1 + (Ce_2 - Ce_1) + (Ce_3 - Ce_2) + \cdots.$$

As a consequence of the inequalities

$$p(e_n e_{n+1} \cdots) \leqq p(e_n) \leqq p(e_n + e_{n+1} + \cdots)$$

the theorem follows for every limit e, since from the two special cases just considered and the relations (7) and (8)

$$p(e) = \lim p(e_n + e_{n+1} + \cdots) = \lim p(e_n e_{n+1} \cdots).$$

It is important to note that the limit of $p(e_n)$ is zero when each e_n contains the succeeding and all the sets together have no point in common. For then the set Ce in equation (18) is exactly the interval ab. The additive property applied to this equation shows that the limit of $p(Ce_n)$ is $p(ab)$, and the relation

$$p(e_n) + p(Ce_n) = p(ab)$$

shows the truth of the statement just made.

Every class $\mathcal{E}$ having the properties described in Theorem 4 contains all sets which are formed from intervals by addition and taking complements, or by repetition of these processes a finite or denumerably infinite number of times. These are the sets to which Borel* first extended the definition of measure, and are called measurable (B). They are important because they are necessarily contained in every class $\mathcal{E}$ which is closed and contains all intervals.

§ 4. *Measurable Functions.*

The definition of a measurable function given in § 1 is adequate for the purposes of the proof there given of the existence of the Lebesgue integral limit for a bounded function. It is unsatisfactory, however, for functions $f(x)$ which at some points have infinite values, because it takes no account of the abscissas x where this happens. It will be evident in the following pages that the application of Lebesgue integrals to non-bounded functions is a feature of the theory comparable in importance with the generalizations of the notions of measure and integration which have already been described, and it is highly desirable, therefore, to modify the definition of § 1 so that it may be more widely applicable. The definition given in this section is equivalent to the earlier one for bounded functions and has the necessary generality.

DEFINITION OF A MEASURABLE FUNCTION.† *Let e be a set of points x at each of which $f(x)$ has either a single finite value, or else one of the values $+\infty$ and $-\infty$. Then $f(x)$ is measurable on e if e is measurable, and if furthermore the set $e[f \geqq a]$ is measurable for every constant a.*

If a bounded function is measurable according to the definition of § 1, it is also measurable according to this definition.

* I, p. 46.

† VII, p. 27; V, vol. 1, 3d edition, p. 69; III, p. 110.

For let M be greater than all the values of $f(x)$ on e, and let b be greater than M. The sets $e[f \geqq a]$ and $e[a \leqq f(x) < b]$ are the same, and the measurability of the latter implies that of the former. Conversely, the equation

$$e[a \leqq f < b] = e[a \leqq f] - e[b \leqq f]$$

shows that a function measurable under the definition of this section is also measurable according to that of § 1.

THEOREM 5. *If for a function $f(x)$ of the kind described above all the sets of any one of the four types*

$$e[f \geqq a], \quad e[f > a], \quad e[f < a], \quad e[f \leqq a]$$

are measurable for every a, then those of the other three have the same property, and f is measurable on e.

Suppose that the sets of the first type are all measurable. It follows that $e[f = a]$ is measurable for every a, since the sum of the sets $e[f \geqq a + 1/n]$ $(n = 1, 2, \cdots)$ is measurable, and $e[f = a]$ is the difference between $e[f \geqq a]$ and this sum. Hence the sets of the second type are also measurable. Furthermore the sum of the first and third sets is the measurable set e, and the fourth is the sum of the third and $e[f = a]$. In a similar manner the measurability of any one of the types implies the measurability of the other three.

COROLLARY. *If $f(x)$ is measurable on e then $e[f = a]$ and $e[a \leqq f < b]$ are measurable for all values of the constants a and b.*

LEMMA 5. If f and φ are both measurable on a set e then $e[f > \varphi]$ is measurable.

For there is always a rational number r between f and φ when $f > \varphi$, and the set $e[f > \varphi]$ is therefore the sum of the denumerable infinity of product sets $e[f > r] \cdot e[r > \varphi]$, where r is a rational number.

THEOREM 6. *If f and φ are both measurable on a set e, then each of the functions f, φ, $f + \varphi$, $f - \varphi$, $f\varphi$, $|f|$, and $1/f$ is measurable on every measurable sub-set of e where it is well-defined, and on the whole of e if an arbitrary constant value is assigned to it at all points where it is not defined.*

If a function f is measurable on e it is also measurable on every measurable sub-set e' of e, since $e'[f \geqq a]$ is the product of e' and $e[f \geqq a]$.

The points where $f \pm \varphi$ is not well-defined are those of the set where the values of f and $\pm \varphi$ are opposite infinities. This set is measurable since $e[f = +\infty]$ is the product of the measurable sets $e[f \geqq n]$ $(n = 1, 2, \cdots)$, and since $e[f = -\infty]$, $e[\varphi = \pm\infty]$ can be similarly shown to be measurable. On the measurable remainder e' of e it follows at once from the definition of a measurable function and Theorem 5 that the measurability of φ implies also that of $\varphi + c$ and $c\varphi$, where c is any constant. Hence the functions $f \pm \varphi$ are measurable on e' since the sets

$$e'[f \pm \varphi > a] = e'[f > a \mp \varphi]$$

are measurable for every a, by Lemma 5. Furthermore f^2 is measurable when f is so, since for every $a \geqq 0$

$$e[f^2 \geqq a] = e[f \geqq \sqrt{a}] + e[f \leqq -\sqrt{a}],$$

and the formula

$$4f\varphi = (f + \varphi)^2 - (f - \varphi)^2$$

shows that $f\varphi$ is measurable on the sub-set of e where f and φ and therefore $f\varphi$ are finite. On the remainder of e the product $f\varphi$ is definitely infinite on sets which are measurable, or else the product of an infinity by zero and indeterminate. The measurability of $|f|$ follows from the measurability of f^2 and the formula

$$e[|f| \geqq a] = e[\sqrt{f^2} \geqq a] = e[f^2 \geqq a^2],$$

which holds also for every $a \geqq 0$. Finally $1/f$ is measurable on the sub-set e' of e where $f \neq 0$, on account of the formulas

$$a > 0: \quad e'\left[\frac{1}{f} \geqq a\right] = e'\left[\frac{1}{a} \geqq f\right] \cdot e'[f > 0],$$

$$a = 0: \quad e'\left[\frac{1}{f} \geqq 0\right] = e'[f > 0],$$

$$a < 0: \quad e'\left[\frac{1}{f} \geqq a\right] = e'[f > 0] + e'\left[f \leqq \frac{1}{a}\right].$$

The last statement of the theorem follows since every set where one of the functions is indeterminate is measurable.

THEOREM 7.* *The upper bound $\varphi(x)$ of a set of functions*

* VII, p. 29; VI, p. 42; V, vol. 1, 3d edition, p. 71.

$f_n(x)$ $(n = 1, 2, \cdots)$ measurable on e, is measurable on e. Similarly the upper and lower limits of a sequence of functions $\{f_n\}$ are measurable. If the sequence has a unique limit, the limit is measurable.

For the set $e[\varphi > a]$ is the sum of the measurable sets $e[f_n > a]$ for the values $n = 1, 2, \cdots$. Furthermore let $\varphi_n(x)$ be the measurable upper bound of the set $f_n, f_{n+1}, \cdots$. The upper limit $L(x)$ of the sequence $\{f_n\}$ is by definition the lower bound of the functions $\varphi_n(x)$, and hence also measurable. The sequence is said to have a unique limit, equal to both the upper and the lower limit, when the upper and lower limits are everywhere the same.

THEOREM 8.* *Let $f(x, y)$ be a function having a single real value at each point of the region $a \leqq x \leqq b$, $0 < y \leqq d$. Suppose furthermore that f is continuous in y for every fixed x, and measurable in x for every fixed y. Then the functions*

$$\varphi(x) = \overline{\lim_{y=0}} f(x, y), \quad \psi(x) = \underline{\lim_{y=0}} f(x, y)$$

are measurable on ab.

Since $f(x, y)$ is continuous in y, the upper bound $\varphi_n(x)$ of the values $f(x, y)$ when x is fixed and $0 < y \leqq 1/n$, is the same as the upper bound of the denumerable set of values $f(x, r)$ where $y = r$ is a rational value of y on the interval $0 < y \leqq 1/n$. Hence by the first part of Theorem 7, $\varphi_n(x)$ is a measurable function of x. Further the function $\varphi(x)$ of the theorem is the limit of $\varphi_n(x)$ as n approaches infinity, and therefore measurable. Similar reasoning holds for $\psi(x)$.

§ 5. *Properties of Lebesgue Integrals of Bounded Functions.*

The integral of Lebesgue has properties similar to those of the classical integral of Riemann, but with two very important extensions. An integral of either type, regarded as a function $I(f, e)$ of the function f to be integrated and of the domain of integration e, has additive properties with respect to each argument similar to those described in the statements 2) and 4) of the theorem below. For the Riemann integral the domain of integration is usually an interval, and the additive property 2) is true for a finite number of them; whereas the additivity of the Lebesgue integral holds even when the domain of integration is decomposed into a de-

* VII, p. 30; VI, p. 442.

numerable infinity of parts. The additive property with respect to the argument f is valid for a Riemann integral when f is the sum of a finite number or an infinite series of bounded integrable functions, provided that the series converges uniformly.* The only requirement prescribed below for this property of a Lebesgue integral is that f shall be the sum of a convergent series of measurable terms f_n for which the partial sums up to n terms have common bounds μ, M. It is one of the most remarkable characteristics of the integrals of Lebesgue that the requirement of uniformity of convergence is unnecessary for the establishment of this property. In § 6 some examples are given to illustrate these generalizations. The property 5) of the theorem is one which both integrals have in common, but the recognition of its validity for the Riemann integral was a consequence of the development of the newer theory. No proof of this property is given in this section because it is a special case of a similar theorem for summable functions which will be proved later.

THEOREM 9.† *The Lebesgue integrals of functions which are bounded and measurable have the following properties:*

1) *If f is bounded and measurable, with bounds μ, M on e, then*

$$\mu m(e) \leqq \int_e f dx \leqq M m(e).$$

The integral is well defined and absolutely continuous as a function $g(e)$ on the measurable sub-sets of e.

2) *If f is bounded and measurable on the sum e of a finite number or an infinite sequence of distinct measurable sets e_n, then*

$$\int_e f dx = \int_{e_1} f dx + \int_{e_2} f dx + \cdots.$$

3) *If f and φ are bounded and measurable, and satisfy $f \leqq \varphi$ on e, then af and $|f|$ are also bounded and measurable on e and*

$$\int_e a f dx = a \int_e f dx, \quad \int_e f dx \leqq \int_e \varphi dx, \quad \left| \int_e f dx \right| \leqq \int_e |f| \, dx.$$

4) *If each of a finite number of functions f_n is bounded and*

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* Hobson, Theory of Functions of a Real Variable, p. 540.

† VII, pp. 39–45; V, vol. 1, 3d edition, pp. 258–259; III, pp. 112–116.

measurable on e, then their sum has the same property and

$$\int_e f dx = \int_e f_1 dx + \int_e f_2 dx + \cdots.$$

The theorem remains true for the sum of a convergent series of functions f_n, provided that the partial sums up to n terms all have absolute values less than a constant M.

5) *If f is bounded and measurable on the interval ab, the function*

$$F(x) = \int_{ax} f dx$$

has f as derivative at all the points of ab except those of a set of measure zero.

The sum S of § 1 whose limit is the Lebesgue integral satisfies the inequalities

$$(\mu - \epsilon)m(e) \leqq S \leqq (M + \epsilon)m(e),$$

since every value η_k having $m(e_k) \neq 0$ lies between $\mu - \epsilon$ and $M + \epsilon$. Hence at the limit as ϵ approaches zero the inequalities of 1) are true. The absolute continuity of the integral as a function of e is an immediate consequence of this mean value theorem.

To prove the property 2) consider first the sum $e = e' + e''$ of two sets on each of which f is bounded and measurable. If S, S', S'' are sums for f on the sets e, e', e'' respectively, formed with the same ladder of values l_k, then $S = S' + S''$ and the additive property for two sets, or indeed a larger finite number of sets, is an easy consequence. The sum e of a sequence $\{e_n\}$ may be written in the form

$$e = e_1 + e_2 + \cdots + e_n + r_n,$$

where $m(r_n)$ approaches zero as n approaches infinity. Hence from the additive property for a finite number of sets

$$\int_e f dx = \int_{e_1} f dx + \cdots + \int_{e_n} f dx + \int_{r_n} f dx,$$

and with the help of the property 1) the last integral is seen to have the limit zero as n increases.

If a sum S is formed for f with values l_k, and a sum S' for af with values al_k, it is easily seen that $S' = aS$, so that the first of the properties 3) also holds when ϵ approaches zero. It is useful to note that a quite similar proof shows that

$$(19) \qquad \int_e (a + f)dx = am(e) + \int_e fdx.$$

To prove the second part of 3), let e be decomposed into the sets $e_k = e[l_{k-1} \leqq f < l_k]$. Then by 2) and the mean value theorem of 1)

$$\int_e \varphi dx = \sum_{k=1}^{n} \int_{e_k} \varphi dx \geqq \sum_{k=1}^{n} l_{k-1} m(e_k).$$

But the last expression is a sum S for the function f, and the property desired is a consequence when ϵ approaches zero. The last part of 3) follows with the help of the first part when the result just obtained is applied to the pairs f, $\varphi = |f|$ and $-f$, $\varphi = |f|$.

The properties 2) and 3) with equation (19) justify the inequalities

$$\int_e (f + \varphi)dx \geqq \sum_{k=1}^{n} \int_{e_k} (l_{k-1} + \varphi)dx = \sum_{k=1}^{n} l_{k-1} m(e_k) + \int_e \varphi dx,$$

$$\int_e (f + \varphi)dx \leqq \sum_{k=1}^{n} \int_{e_k} (l_k + \varphi)dx = \sum_{k=1}^{n} l_k m(e_k) + \int_e \varphi dx,$$

from which it follows at the limit that

$$\int_e (f + \varphi)dx = \int_e fdx + \int_e \varphi dx.$$

This establishes the property 4) for every sum of a finite number of measurable functions. To prove that a similar property holds for series the following lemma is useful.

LEMMA 6. *If a sequence $s_n(x)$ of functions measurable on e converges to a bounded limit f, then for every ϵ the sets $e_n = e[|f - s_n| > \epsilon]$ are such that $\lim_{n=\infty} m(e_n) = 0$.*

The complete limit c of the sets e_n can contain no point, since a value x in c would be in an infinity of sets e_n, and therefore $|f - s_n|$ would exceed ϵ for an infinity of values of n. This is not possible if f is the limit of the sequence $\{f_n\}$ at x. It follows therefore from formula (8) and the corollary to Theorem 4 that

$$m(c) = \lim m(e_n + e_{n+1} + \cdots) = 0,$$

and hence that $\lim m(e_n) = 0$.

It is now possible to prove the additive property 4) for an infinite series of functions f_n. For let s_n be the sum of the first n terms of the series, and e_n the set described in the lemma. Then with the help of 3), 4), 2), and 1),

$$\left| \int_e f dx - \int_e s_n dx \right| \leqq \left| \int_{e-e_n} (f - s_n) dx \right| + \left| \int_{e_n} (f - s_n) dx \right|$$
$$\leqq \epsilon m(e) + 2Mm(e_n),$$

and this can be made arbitrarily small by taking ϵ sufficiently small and n sufficiently large.

COROLLARY. *If a sequence $\{s_n\}$ of functions measurable on e has a limit f, and all the elements of the sequence have the same bounds μ, M, then*

$$\lim_{n=\infty} \int_e s_n dx = \int_e f dx.$$

§ 6. *Examples of Measurable Sets and Functions.*

The definition of measure commonly accepted before the ideas of Borel and his followers were developed was the following well-known one of Jordan.* Let the interval ab be divided into a finite number of sub-intervals of norm δ, and let s, S be the sums of the intervals entirely in e, and of the intervals containing points of e but not necessarily entirely in e, respectively. Then s and S have limits as δ approaches zero which are called the interior and exterior measures σ, Σ of the set e. If the two are equal the set e is said to be measurable (J), and its Jordan measure is the common value $\sigma = \Sigma$.

It is provable that the totality $\mathfrak{F}$ of sets measurable (J), and the Jordan measure function, have properties similar to those described in Theorem 4, the principal and very important difference being that $\mathfrak{F}$ is additively closed, and the Jordan measure function is additive, for a finite number of sets only. A very simple example can be given to illustrate the greater power of the new definition of measure in this respect. A set consisting of a single point has measure zero according to either definition. It is provable directly from the definition,

* IX, p. 28.

as will be indicated just below, or from the additive property of the class $\mathcal{E}$ described in Theorem 4, that the set of rational points on the interval $0 \leqq x \leqq 1$, which is denumerable, must have Borel-Lebesgue measure zero. But it is evident also that the Jordan measure of this set is not well defined, since the exterior measure of the set is 1 and the interior measure 0.

It is reasonable to ask whether or not all sets have measure according to the definition of § 3, but it seems impossible to give a conclusive answer to the question at present. Sets not measurable have been constructed by Van Vleck and Lebesgue with the help of principles from the theory of aggregates which have been in dispute, and which so far remain unproved.

THEOREM 10. *Every denumerable set of points has measure zero according to the definition of measure of* § 3. *Further every closed set of points is measurable* (B).

If the elements of the denumerable set are denoted by x_n ($n = 1, 2, \cdots$), each x_n can be enclosed in an interval of length $\epsilon/2^n$, and the sum of the lengths of these intervals is ϵ. Consequently the measure of the set is zero.

A point which is exterior to a closed set e is enclosable in an interval containing no point of e in its interior, but whose end points are in e. The totality of such intervals is denumerable, since there is but a finite number of them with lengths greater than $1/n$. The complement of e is therefore this measurable set consisting of a denumerable set of intervals $\Sigma\alpha_n$, which implies that e is measurable (B) also.

The Riemann integral of a bounded function on an interval ab is a notion closely allied to that of Jordan measure. For let ab be divided into a finite number of intervals Δx_k ($k = 1, 2, \cdots, n$) of norm δ, and let m_k, M_k be the lower and upper bounds of the values of $f(x)$ in Δx_k. The two sums

$$s = \sum_{k=1}^{n} m_k \Delta x_k, \quad S = \sum_{k=1}^{n} M_k \Delta x_k$$

have limits σ, Σ as δ approaches zero, and when the two are equal the function $f(x)$ is said to be integrable (R) on ab, the value of the integral being the common limit $\sigma = \Sigma$. If $f(x) = 1$ at every point of a set e and $f(x) = 0$ elsewhere, the limits σ, Σ are exactly the Jordan interior and exterior measures of e, so that the existence of the integral of this particular function, and the measurability of e according to the Jordan definition, imply each other.

A necessary and sufficient condition that $f(x)$ be integrable (R) is that

$$\lim_{\delta=0} (S - s) = 0.* \tag{20}$$

Lebesgue† has transformed this condition in a most interesting and useful way, as indicated in the following theorem.

THEOREM 11. *A necessary and sufficient condition that a bounded function $f(x)$ be integrable in the sense of Riemann on ab, is that the set of discontinuities of $f(x)$ have measure zero.*

The proof of this theorem by Lebesgue made no use of the properties of Borel-Lebesgue measure except the concept of a set of measure zero. A proof of Theorem 12 below which has recently been given by de la Vallée Poussin‡ makes use of reasoning which, with the help of the theorems of the preceding section, leads more directly to the desired result. Let $M(x)$ be the limit as δ approaches zero of the maximum of f on an interval of norm δ with x as interior point. Furthermore let $\Phi(x)$ be a function having the value M_k at interior points of each interval Δx_k of a partition of ab, and having an arbitrarily assigned value, say zero, at the division points of the partition. Then

$$S = \int_{ab} \Phi(x)dx,$$

where the integral is to be interpreted in the sense of Lebesgue. Consider now a sequence $\{S_n\}$ of sums with norms approaching zero, and let $\{\Phi_n\}$ be the corresponding sequence of functions Φ. Every element of this last sequence is measurable, and hence its upper and lower limits are measurable, by Theorem 7. Furthermore the totality of partition points of the sums S_n forms a denumerable set of measure zero, and at every other point the sequence $\{\Phi_n\}$ is easily seen to have the unique limit $M(x)$. It follows that $M(x)$ is also measurable, since it is identical with a measurable function except on a set of measure zero; and from the Corollary to Theorem 9

$$\int_{ab} M(x)dx = \lim_{n=\infty} \int_{ab} \Phi_n(x)dx = \lim_{\delta=0} S = \Sigma,$$

* V, vol. 1, 3d edition, p. 255.
† III, pp. 29, 109.
‡ VII, p. 55.

since the interval ab differs from the set of points in which $\lim \Phi_n = M(x)$ only by a set of measure zero. A precisely similar argument gives σ as the integral of the function $m(x)$ which is the limit as δ approaches zero of the minimum of f on an interval of length δ having x as interior point. Hence a necessary and sufficient condition for f to be integrable (R) is

$$\int_{ab} [M(x) - m(x)]dx = 0.$$

The integrand of this integral is everywhere positive or zero, and when the integral vanishes can exceed $1/n$ only on a set of points e_n of measure zero. Otherwise, by the property 1) of Theorem 9, the integral would be different from zero. Hence a necessary and sufficient condition that the integral shall vanish is that the set of discontinuities of f, which is the totality of points where $M(x) - m(x) > 0$ and identical with the sum of the sets e_n, should have measure zero.

THEOREM 12.* *A bounded function $f(x)$ which is integrable in the sense of Riemann on ab is also measurable on ab, and its Riemann and Lebesgue integrals are equal. In particular for a continuous function the sets $e[f \geqq \alpha]$ are measurable (B) for every α.*

The set of points $e[f \geqq \alpha]$ contains all of its limit points at which f is continuous. The other limit points are discontinuities of f and therefore form a set of measure zero, by Theorem 11. Hence $e[f \geqq \alpha]$ plus a set of measure zero is closed and therefore measurable, from which it follows that $e[f \geqq \alpha]$ itself is measurable. For the case of a continuous function this set is closed and therefore measurable (B). From the properties 1) and 2) of Theorem 9 the Lebesgue integral of f satisfies the inequalities

$$\sum_{k=1}^{n} m_k \Delta x_k \leqq \sum_{k=1}^{n} \int_{\Delta x_k} f dx \leqq \sum_{k=1}^{n} M_k \Delta x_k$$

for every partition of ab into intervals Δx_k of norm δ. But the first and last members of this inequality are the sums s, S whose limits as δ approaches zero are both equal to the Riemann integral of f. Hence the equality of the two integrals of f on ab is proved.

* VII, p. 55; V, vol. 1, 3d edition, p. 259.

COROLLARY. *Every set e which is measurable in the sense of Jordan is also measurable according to the definition of § 3, and the two measures are equal.*

Let $f(x)$ be the function described above which has the value $f(x) = 1$ at all points of e, but which is zero elsewhere. If the Jordan measure of e is well defined it is equal to the Riemann integral of $f(x)$ on ab, and hence also equal to the Lebesgue integral. But the latter is the Borel-Lebesgue measure of e, so that the two measures are equal.

It may be of interest to consider for a moment a simple example of a function integrable according to Lebesgue, but not so according to Riemann. Let $f(x) = 2$ on the set e_1 of irrational points on the interval $0 \leqq x \leqq 1$, and let $f(x) = 1$ on the set e_2 where x is rational. This function has sums $s = 1$, $S = 2$ for every subdivision of the interval ab into sub-intervals Δx_k, and hence has no Riemann integral. On the other hand the Lebesgue integral has the well-defined value

$$\int_{01} f dx = \int_{e_1} f dx + \int_{e_2} f dx = 2.$$

THEOREM 13. *Let $\alpha(y)$ be the measure of the set $e[\mu \leqq f < y]$ for a function $f(x)$ measurable and with $\mu < f < M$ on e. Then $\alpha(y)$ is a monotonically increasing function and the Lebesgue integral of f is expressible in the forms*

$$\int_e f dx = \int_\mu^M y d\alpha = Mm(e) - \int_\mu^M \alpha(y) dy, \tag{21}$$

where the second integral is to be taken in the sense of Stieltjes described below, and the last is an integral of Riemann.

If $\varphi(y)$ and $\alpha(y)$ are two single-valued functions on the interval $\mu \leqq y \leqq M$, the corresponding Stieltjes integral limit is by definition

$$\int_\mu^M \varphi d\alpha = \lim_{\epsilon = 0} \sum_{k=1}^{n} \varphi(\eta_k)[\alpha(l_k) - \alpha(l_{k-1})],$$

where the values l_k $(k = 0, 1, \cdots, n)$ with $l_0 = \mu$, $l_n = M$ define a partition of μM into sub-intervals of norm ϵ having $l_{k-1} \leqq l_k$, $l_{k-1} \leqq \eta_k \leqq l_k$. It is useful, as will be seen presently, to permit successive values l_{k-1}, l_k to be equal. The existence of the limit as thus defined is provable* for every

* See, for example, Riesz, loc. cit., p. 37; Stieltjes, loc. cit., p. J. 71.

pair of functions φ, α such that φ is continuous and α of limited variation on μM, and it is an immediate consequence of this definition and that of § 1 that the first of the equations (21) is true, since $m(e_k)$ in (3) is exactly the difference $\alpha(l_k) - \alpha(l_{k-1})$. The choice $l_0 = \mu$, $l_n = M$ for the sum in (3) is now permissible since by the hypothesis here used $\mu < f < M$ on e.

Suppose now that the Stieltjes integral exists for two functions φ, α. Then it also exists when the rôles of φ and α are interchanged, and furthermore*

$$\int_\mu^M \varphi d\alpha + \int_\mu^M \alpha d\varphi = \varphi(M)\alpha(M) - \varphi(\mu)\alpha(\mu). \tag{22}$$

For let the values η_k ($k = 1, 2, \cdots, n$; $\eta_1 = \mu$, $\eta_n = M$) define a partition of μM of norm $\epsilon/2$, and let l_k ($k = 1, 2, \cdots, n-1$) be chosen for each sub-interval so that $\eta_k \leqq l_k \leqq \eta_{k+1}$. Then the values l_k with the addition of $l_0 = \mu$, $l_n = M$ form a partition of μM with norm ϵ and are such that $l_{k-1} \leqq l_k$, $l_{k-1} \leqq \eta_k \leqq l_k$. It is a matter of algebra only to show now that

$$\sum_{k=2}^{n} \alpha(l_{k-1})[\varphi(\eta_k) - \varphi(\eta_{k-1})] = - \sum_{k=1}^{n} \varphi(\eta_k)[\alpha(l_k) - \alpha(l_{k-1})] + \varphi(M)\alpha(M) - \varphi(\mu)\alpha(\mu).$$

Since by hypothesis the limit of the sum on the right exists, the same is true of that on the left, and the formula (22) is the limit of this when ϵ approaches zero. Furthermore formula (22) justifies easily the second of the expressions (21) when $\varphi = y$, since $\alpha(M) = m(e)$, $\alpha(\mu) = 0$.

THEOREM 14. *Every derivative number Λ of a function $f(x)$ continuous on ab is measurable, and the sets $e[\Lambda \geqq \alpha]$ are all measurable (B).*

If the definition of $f(x)$ is extended so that $f(x) = f(b)$ for $x > b$, the resulting function is continuous for $x \geqq a$ and its forward derivative numbers are by definition the upper and lower limits of the quotient

$$g(x, h) = \frac{f(x+h) - f(x)}{h} \tag{23}$$

as h approaches zero over positive values. The quotient is a

* Riesz, loc. cit., p. 37; Stieltjes, loc. cit., p. J. 72.

continuous function of both x and h for $a \leqq x \leqq b, 0 < h$, and the fact that its limits are measurable (B) is a consequence of the last statement in Theorem 12, and the proofs of Theorems 7 and 8. A similar argument holds for the two backward derivatives.

§ 7. *Summable Functions.*

It has been remarked in § 4 that a part of the effectiveness of the Lebesgue theory of integration is associated with its applications to functions which are not necessarily bounded. This is due to the fact that the upper and lower limits of every sequence $\{f_k\}$ of measurable functions are measurable, but they may have $+\infty$ or $-\infty$ as functional values as well as finite ones. Consider then a function f which is measurable according to the definition of § 4, and let it be decomposed in the form

$$f = \frac{|f| + f}{2} - \frac{|f| - f}{2} = \varphi - \psi, \tag{24}$$

with $\varphi = 0$ when $f = -\infty$ and $\psi = 0$ when $f = +\infty$. Each of the functions φ, ψ is $\geqq 0$, and measurable according to Theorem 6. Consider now the function φ_n defined by the conditions

$$\begin{aligned} \varphi_n &= \varphi \text{ when } \varphi \leqq n, \\ \varphi_n &= n \text{ when } \varphi > n. \end{aligned} \tag{25}$$

This function is also measurable, as one may readily verify.

DEFINITION OF A SUMMABLE FUNCTION AND ITS INTEGRAL.* *A function $\varphi(x) \geqq 0$ is summable on e if it is measurable on e and such that the limit*

$$\int_e \varphi dx = \lim_{n=+\infty} \int_e \varphi_n dx$$

exists. A function f of arbitrary sign is said to be summable on e if the functions φ and ψ of equation (24) *are both summable on e. The integral of f is then defined to be*

$$\int_e f dx = \int_e \varphi dx - \int_e \psi dx. \tag{26}$$

* VII, p. 45; VI, p. 444; V, vol. 1, 3d edition, p. 260; III, p. 115.

It follows at once from this definition and the mean value theorem that if f is summable the set of values x where $f(x)$ is $+\infty$ or $-\infty$ has measure zero. When e is replaced by the interval ax the integral on the left in (26) becomes a function of x expressed as a difference of two increasing functions, and hence is of limited variation. Further properties of integrals of summable functions are summarized in the following theorem, the proof of property 5) being again postponed to § 9.

THEOREM 15. *If in the statement of Theorem* 9 *the words "bounded and measurable" are everywhere replaced by the word summable, the properties* 1)–5) *are still true, the only exception being that the additive property* 4) *does not necessarily hold for a denumerable infinity of functions.*

The proofs will be made first for functions which are nowhere negative. The law of the mean in 1) for summable functions is a consequence of the second formula in 3) which will be proved later. If $f(x)$ is summable on e it is so on every measurable subset of e, as one may see at once from the definition. The absolute continuity of the integral of a non-negative summable function follows, since for a sufficiently large value of n

$$\int_e \varphi dx \leqq \int_e \varphi_n dx + \frac{\epsilon}{2} \leqq nm(e) + \frac{\epsilon}{2},$$

and when $m(e)$ is sufficiently small this is less than ϵ.

If φ is non-negative and summable on a sum e of measurable sets e_k $(k = 1, 2, \cdots)$, then

$$\int_e \varphi_n dx \geqq \sum_{k=1}^{p} \int_{e_k} \varphi_n dx,$$

$$\int_e \varphi_n dx = \sum_{k=1}^{\infty} \int_{e_k} \varphi_n dx \leqq \sum_{k=1}^{\infty} \int_{e_k} \varphi dx.$$

By letting n and then p approach infinity in the first of these, and by letting n approach infinity in the second, one obtains the inequalities

$$\int_e \varphi dx \geqq \sum_{k=1}^{\infty} \int_{e_k} \varphi dx, \quad \int_e \varphi dx \leqq \sum_{k=1}^{\infty} \int_{e_k} \varphi dx,$$

which prove property 2).

The first and second relations under 3) follow readily from the definition of a summable function when it is noted that for non-negative functions and $a > 0$

$$(af)_{an} = af_n, \quad f_n \leqq \varphi_n. \tag{27}$$

The argument for $a < 0$ is similar. The third relation of 3) is evident for positive functions since then $|f| = f$.

The proof here given for the second of the properties 3) justifies more than is stated in the theorem. For if φ is positive and summable and $0 \leqq f \leqq \varphi$ it is a consequence of the second of the relations (27) that f must also be summable.

If the functions f, g are non-negative and $h = f + g$ one may verify the relation

$$h_n \leqq f_n + g_n \leqq h_{2n}.$$

The integration of this shows that when f and g are summable, the same is true of h, and at the limit when n approaches infinity

$$\int_e hdx \leqq \int_e fdx + \int_e gdx \leqq \int_e hdx.$$

Hence the additive property 4) holds for a finite number of non-negative functions.

For a function f of arbitrary sign the absolute continuity of 1), the additive property 2), and the first relation of 3), follow from the preceding proofs when f is decomposed as in equation (24). The property 4) can now be derived for the sum $h = f + g$ of two summable functions of arbitrary sign by subdividing e into sub-regions where no one of f, g, h changes sign. In each of these regions h is summable, since it does not change sign and is numerically less than or equal to a non-negative summable function, and the property 4) is a consequence of the proof for non-negative functions, provided that the functions in the relation $h = f + g$ are suitably transposed. Addition of these results shows that the same is true for the original region, since a function summable on each of a finite number of sets is readily seen to be summable on their sum, and since in that case the additive property 2) holds. The second of the relations 3) is a consequence of 4) and the first of the relations 3), since $\varphi - f \geqq 0$ and

$$\int_e \varphi dx - \int_e fdx = \int_e (\varphi - f)dx \geqq 0.$$

If f is decomposed as in equation (24) the value of $|f|$ is $\varphi + \psi$ which by 4) is summable and such that

$$\int_e |f|\, dx = \int_e \varphi dx + \int_e \psi dx \geqq \int_e \varphi dx - \int_e \psi dx = \int_e f dx.$$

One can also state conversely that $|f|$ summable implies that φ and ψ, and consequently f, are summable. Finally the mean value theorem is an immediate consequence of the second of the relations 3). In the statement of this theorem it is understood that one or both of the values μ, M may be infinite.

The notions of major and minor functions of a summable function $f(x)$ were devised by de la Vallée Poussin and described by him as being very precious in the theory of Lebesgue integrals. He has certainly applied them with great success.

DEFINITION OF MAJOR AND MINOR FUNCTIONS OF A SUMMABLE FUNCTION f.* *A function $\Phi(x)$ is a major function for a function f summable on ab if for some $\epsilon > 0$ it has the properties*

1) $$\int_{ax} f dx \leqq \Phi(x) - \Phi(a) < \int_{ax} f dx + \epsilon;$$

2) *every derivative number Λ of Φ satisfies the inequality $\Lambda > f$ at every value x where $f(x) \neq +\infty$.*

The same definition characterizes a minor function $\varphi(x)$ if the signs before ϵ and ∞ are changed and the inequalities in 1) *and* 2) *are inverted.*

The applicability of these functions is a consequence of the following theorem:

THEOREM 16. *There exists a major function $\Phi(x)$ for every function $f(x)$ summable on ab and every constant $\epsilon > 0$. A similar statement holds for minor functions.*

To prove this consider first a function $f(x)$ which is summable and non-negative on ab, with a ladder of values l_k [$k = 0, 1, \cdots$; $l_0 = 0, 0 < l_k - l_{k-1} < \epsilon/2(b - a)$]. Let the part e_k of ab where $l_{k-1} \leqq f < l_k$ be enclosed in the interior of a denumerable set A_k of non-overlapping intervals having $m(A_k - e_k) < \epsilon_k$, the constants ϵ_k being so chosen that $\Sigma l_k \epsilon_k < \epsilon/2$. If the portions of e_k and A_k on ax, and also their measures, are denoted by $e_k(x)$ and $A_k(x)$, respectively, then

* V, vol. 1, 3d edition, p. 269; VII, p. 74; VI, p. 461.

$$\Phi(x) = \sum_k l_k A_k(x) \geqq \sum_k l_k e_k(x) \geqq \sum_k \int_{e_k(x)} f dx = \int_{ax} f dx.$$

Furthermore

$$\sum_k l_k A_k(x) - \sum_k l_k e_k(x) < \sum_k l_k \epsilon_k < \frac{\epsilon}{2},$$

$$\sum_k l_k e_k(x) - \int_{ax} f dx = \sum_k \int_{e_k(x)} (l_k - f) dx < \frac{\epsilon}{2},$$

so that by adding these two relations the function $\Phi(x)$ is seen to satisfy the condition 1) of the definition. But every value x where $f(x)$ is finite is in one of the sets e_k and therefore interior to the corresponding A_k. Consequently the term $l_k A_k(x)$ has derivative l_k at x, and since all the other terms of Φ are non-decreasing functions, it follows that all derivative numbers of Φ are at least equal to $l_k > f$.

For a function f of the form (24) a constant n can be selected so large that

$$\int_{ax} \varphi dx - \int_{ax} \psi_n dx - \int_{ax} f dx < \frac{\epsilon}{2}.$$

A major function for $\varphi - \psi_n$ with constant $\epsilon/2$ will therefore be a major function for f with constant ϵ. But $\varphi - \psi_n + n$ is non-negative and has a major function $\Phi(x)$ for $\epsilon/2$, and the function $\Phi(x) - nx$ plays the same rôle for $\varphi - \psi_n$.

A major function of $-f$ is the negative of a minor function for f, so that the theorem is proved in its entirety.

§ 8. *A Generalization of the Fundamental Theorem of Integral Calculus.**

A function $G(x)$ of limited variation and continuous on ab may be defined as one expressible in the form

$$G(x) = P_1(x) - P_2(x), \tag{28}$$

where P_1 and P_2 are continuous and monotonically increasing. If the process of §§ 2, 3 is applied to P_1 and P_2, two classes of point sets $\mathcal{E}_1$, $\mathcal{E}_2$ and two functions $p_1(e)$, $p_2(e)$ are defined having the relationships described in Theorem 4. The greatest common subclass $\mathcal{E}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ contains all intervals on ab

* VII, Chapitre VI; VI, § 8; IV.

and is closed. Furthermore the function

$$g(e) = p_1(e) - p_2(e)$$

is continuous and additive on $\mathcal{E}$, and such that for every set e of $\mathcal{E}$ and every constant $\epsilon > 0$ there exists a sum A of intervals enclosing e such that

$$p_1(A - e) + p_2(A - e) < \epsilon. \tag{29}$$

The functions $g(e)$ defined in this way are of a very general type which is the subject of the following theorems.

THEOREM 17. *Consider a function $g(e)$ which is continuous and additive on a closed class of point sets $\mathcal{E}$ containing all the intervals on ab. On the totality of subsets in $\mathcal{E}$ of an element e of $\mathcal{E}$ the function $g(e)$ has a positive maximum $p_1(e)$, and $g(e)$ is expressible as the difference of two non-negative functions*

$$g(e) = p_1(e) - [p_1(e) - g(e)] = p_1(e) - p_2(e).$$

The functions $p_1(e)$ and $p_2(e)$ are continuous and additive on $\mathcal{E}$. The sum

$$T(e) = p_1(e) + p_2(e)$$

is called the total variation of g.

Since $g(e)$ is continuous it is clear that its least upper bound on the subsets of a set e_1 must be positive or zero, if such a bound exists. If there were no such upper bound on e_1, one could select a subset e_2 so that $g(e_2)$ and $g(e_1 - e_2)$ are both numerically greater than unity, and so that g would also have no upper bound on e_2. By repeating this process a sequence $\{e_k\}$ would be found, with each element contained in the preceding, and having a greatest common subset e such that

$$e_1 = e + (e_1 - e_2) + (e_2 - e_3) + \cdots.$$

This would contradict the additivity of g, however, since each term after the first in the sum

$$g(e) + g(e_1 - e_2) + g(e_2 - e_3) + \cdots$$

would numerically be greater than unity, and its sum could not be $g(e_1)$.

The upper bound $p_1(e)$ is continuous since $g(e)$ is so, and it is also additive. For if the sets e_k are distinct and

$$e = e_1 + e_2 + \cdots,$$

one may cause g to vary on subsets of e so that it approaches $p_1(e)$. On the corresponding subsets of e_k the values of g will never exceed $p_1(e_k)$, and it follows from the additivity of g that

$$p_1(e) \leqq p_1(e_1) + p_1(e_2) + \cdots.$$

Furthermore if g varies on subsets of e_k so that its values on all the e_k's approach simultaneously the upper bounds $p_1(e_k)$, then it follows that

$$p_1(e) \geqq p_1(e_1) + p_1(e_2) + \cdots,$$

and the additivity of $p_1(e)$ is proved.

DEFINITION. *A class $\mathcal{E}$ of point sets is said to be a normal class for a function $g(e)$ if it has the properties*

1) *$\mathcal{E}$ is a closed class of measurable sets containing all the intervals on ab;*

2) *$g(e)$ is continuous and additive on $\mathcal{E}$;*

3) *for every element e of $\mathcal{E}$ and every constant $\epsilon > 0$ there exists a sum of intervals A enclosing e such that $T(A - e) < \epsilon$.*

A set A can be chosen satisfying condition 3) and at the same time having $m(A - e) < \epsilon$. For if A_1 satisfies 3), and A_2 is such that $m(A_2 - e) < \epsilon$, the product set $A = A_1A_2$ will have the two properties desired, since $T(e)$ and $m(e)$ are both positive functions. When the property 3) holds $g(e)$ is the limit of the values of g on sets e' enclosing e, in the sense that for every e and ϵ there exists a set A enclosing e such that

$$|\, g(e') - g(e) \,| \leqq |\, T(e') - T(e) \,| \leqq T(A - e) < \epsilon$$

for every e' in A and containing e. This is a consequence of the fact that $T(e)$ is non-negative and additive.

The function $p(e)$ of § 2 generated by a monotonically increasing continuous function $P(x)$ is an example of a function on a normal class $\mathcal{E}$, as one may see by extracting from the class $\mathcal{E}$ of Theorem 4 all except its measurable sets. The following theorem shows how the most general function $g(e)$ on a normal class is always expressible in terms of functions of this type.

THEOREM 18. *For every function $G(x)$ of limited variation the functions $P_1(x)$, $P_2(x)$ of equation* (28) *define as in* §§ 2, 3 *two functions $p_1(e)$, $p_2(e)$ with normal classes $\mathcal{E}_1$, $\mathcal{E}_2$. The* *totality $\mathcal{E}$ of measurable sets common to $\mathcal{E}_1$ and $\mathcal{E}_2$ is a normal class for the function*

$$g(e) = p_1(e) - p_2(e). \tag{30}$$

Conversely, every function $g(e)$ on a normal class $\mathcal{E}$ is identical on $\mathcal{E}$ with a function $g(e)$ generated in this way by a continuous function $G(x)$ of limited variation.

The first part of the theorem is a consequence of the remarks in the first paragraph of this section, except the statement that $\mathcal{E}$ is a normal class for $g(e)$. $\mathcal{E}$ is necessarily closed and contains all intervals, since it is the greatest common subclass of three classes with these properties. Furthermore, the property 3) of the definition follows at once from the inequality (29), since for the function (30) the total variation $T(e)$ is surely less than $p_1(e) + p_2(e)$.

If $g(e)$ has the normal class $\mathcal{E}$ then the function $p_1(e)$ of Theorem 17 is the lower bound of the values $p_1(A)$ on sets of intervals A enclosing e, on account of the property 3) of the definition of a normal class, and is identical with the values $\overline{p}_1(e)$ defined as in § 3 by the function

$$P_1(x) = p_1(\omega) \qquad (\omega = \text{the interval } ax).$$

The identity of p_1 and $\overline{p}_1$ on intervals is evident with the help of the additive property of p_1, and it persists for other sets e because $p_1(e)$ and $\overline{p}_1(e)$ are the lower bounds of p_1 and $\overline{p}_1$, respectively, on sets A of intervals containing e. Furthermore the relation (14) is true for every element of $\mathcal{E}$, since $\overline{p}_1 = p_1$ is additive, and it follows that $\mathcal{E}$ is necessarily contained in the totality of sets on which the exterior and interior values $\overline{p}_1(e)$, $\underline{p}_1(e)$ defined by $P_1(x)$ are identical. Similar remarks hold for $p_2(e)$, and the theorem is therefore proved.

DEFINITION. *The derivative numbers of a function $g(e)$ on a normal class $\mathcal{E}$ are the four derivative numbers of the function*

$$G(x) = g(\omega) \qquad (\omega = \text{the interval } ax). \tag{31}$$

The singularities of $g(e)$ are the points where all four of its derivatives are $+\infty$, or all four $-\infty$.

With these preliminary notions and theorems at hand it is possible to proceed to the proof of a series of lemmas which lead to the theorem which is the object of this section.

LEMMA 7. The derivative numbers of $g(e)$ are all summable on ab, and the set E of singularities of $g(e)$ has therefore measure zero. Futhermore it belongs to $\mathcal{E}$.

The derivative numbers of $g(e)$ are measurable, by Theorem 14, since $G(x)$ in equation (31) is continuous. The further proof can be made for the upper forward derivative Λ of the positive function $T(e)$ since the derivative numbers of $g(e)$ all have absolute values less than Λ. Let Λ_n be constructed from Λ as in (25), and let $\varphi(x)$ be one of the minor functions of Λ_n with constant ϵ. The function $T(\omega) - \varphi(x)$ has its upper forward derivative greater than or equal to zero, since every derivative number of φ is less than Λ. Consequently $T(ab) \geqq \varphi(b) - \varphi(a)$,* and by letting ϵ approach zero it follows that

$$T(ab) \geqq \int_{ab} \Lambda_n dx,$$

and Λ is seen to be summable. The set of points where $\Lambda = +\infty$ includes all the points of E, and has measure zero since Λ is summable.

Since the sets $e[\Lambda \geqq n]$, for every derivative number Λ of $g(e)$ and every positive integer n, are all measurable (B) by Theorem 14, and since their product is the set of points where all four derivative numbers of $g(e)$ are $+\infty$, it follows that E also is measurable (B) and belongs to $\mathcal{E}$.

LEMMA 8. *On a set e where a derivative number Λ of $g(e)$ is never $-\infty$ it is true that*

$$g(e) \geqq \int_e \Lambda dx. \tag{32}$$

The inequality can be inverted if $\Lambda \neq +\infty$ on e.

The proof will first be given for an interval ω with end points α, β. Let $\varphi(x)$ be a minor function with constant ϵ for the function Λ on the interval ω. Then the function

$$\psi(x) = g(\alpha x) - \varphi(x)$$

has its upper derivative $D\psi$, of the same type (forward or backward) as Λ, positive or zero, since

$$\Lambda \leqq D\varphi + D\psi, \quad \Lambda > D\varphi.$$

Hence $g(\omega) \geqq \varphi(\beta) - \varphi(\alpha)$, and the inequality of the lemma for $e = \omega$ is a consequence when ϵ approaches zero.

* V, vol. 1, 3d edition, p. 99.

If a set of intervals A encloses in its interior all the points where $\Lambda = -\infty$ then the inequality (32) is true on $e = CA$. For in the first place the function

$$\psi(e) = g(e \cdot CA) + T(e \cdot A) \geqq g(e)$$

has $\mathcal{E}$ as a normal class. The conditions 1), 2) of the definition of a normal class are clearly satisfied, and 3) also holds since for every denumerable set of intervals B enclosing e the total variation T_ψ of ψ satisfies the relation

$$T_\psi(B - e) \leqq T([B - e] \cdot CA) + T([B - e] \cdot A) = T(B - e).$$

Furthermore the derivative number Λ_ψ of the same type as Λ is greater than or equal to Λ at every point x, and never takes the value $-\infty$, since this can happen to Λ only at points interior to A in the neighborhood of which $\psi(e) = T(e) \geqq 0$. Hence on every interval ω whatsoever

$$\psi(\omega) \geqq \int_\omega \Lambda_\psi dx \geqq \int_\omega \Lambda dx.$$

But this must be true also for every set e of $\mathcal{E}$, since the values of ψ and the last integral on such a set are limits of their values on sets B of intervals enclosing e, according to the remarks following the definition given above of a normal class. In particular for $e = CA$

$$\psi(CA) = g(CA) \geqq \int_{CA} \Lambda dx.$$

Finally the inequality (32) is true for every set e whatsoever on which $\Lambda \neq -\infty$, since a set A enclosing Ce encloses all points where $\Lambda = -\infty$, and

$$| g(CA) - g(e) | = | g(Ce) - g(A) | < \epsilon,$$

$$\left| \int_{CA} \Lambda dx - \int_e \Lambda dx \right| = \left| \int_{Ce} \Lambda dx - \int_A \Lambda dx \right| < \epsilon,$$

provided that A is chosen enclosing Ce in its interior and so that $g(A - Ce)$ and $m(A - Ce)$ are both sufficiently small.

Corollary 1. If a derivative Λ of $g(e)$ is finite valued at every point of e then

$$g(e) = \int_e \Lambda dx.$$

COROLLARY 2. The value of $g(e)$ is zero on every set e of measure zero on which a derivative number of g is everywhere finite valued, or on which two derivatives have everywhere the values $+\infty$ and $-\infty$, respectively.

The first corollary is true since on the set e there described the inequality (32) and its inverse are both justified by the lemma; and the first part of the second corollary follows at once since the integral vanishes when the measure of e is zero. For a set e satisfying the condition in the last part of Corollary 2 the inequality (32) holds with opposite senses for the two derivatives, and the integrals on the right are both zero.

THEOREM 19.* *For every derivative number* Λ *of a function* $g(e)$ *on a normal class* $\mathcal{E}$, *the equation*

$$g(e) = g(eE) + \int_e \Lambda dx \tag{33}$$

holds, where e *is an arbitrary element of* $\mathcal{E}$ *and* E *is the set of singularities of* g. *If a similar equation holds for a second set* E' *of measure zero and another function* Λ', *then* $g(eE) = g(eE')$ *for every* e, *and* Λ *coincides with* Λ' *almost everywhere, i. e., except possibly on a set of measure zero.*

For let E_1 be the totality of points where Λ is infinite. Then

$$g(e) = g(eE_1) + g(eCE_1) = g(eE_1) + \int_e \Lambda dx$$

because of Corollary 1 to Lemma 8 and the fact that E_1 has measure zero. Furthermore $g(eE_1) = g(eE)$, by Corollary 2 to Lemma 8, since $E_1 - E$ is of measure zero and can be subdivided into sets on each of which either one of the derivatives of g is finite valued, or else two have everywhere the values $+\infty$ and $-\infty$, respectively. Hence the first part of the theorem is true.

If equation (33) holds for a second pair E', Λ' then

$$g(eE) + \int_e \Lambda dx = g(eE') + \int_e \Lambda' dx.$$

After replacing e by $eE + eE'$ it follows that the first two terms on the two sides are equal, and hence also the integrals, for every e. If the integrals are equal an application of the mean value theorem shows that the set of points where

* VII, p. 93; VI, p. 476; V, vol. 1, 3d edition, p. 277.

$\Lambda - \Lambda' > 1/n$ must have measure zero for every positive integer n. The sum of these sets has therefore measure zero, and it includes all the points where $\Lambda - \Lambda'$ is positive. A similar argument holds for the points where $\Lambda - \Lambda'$ is negative.

COROLLARY 1. *The function g has all of its derivative numbers equal, and hence has a unique derivative, almost everywhere.*

For the equation (33) holds for all four of the derivative numbers of f, and it follows from the last statement of the theorem that they must be equal except on a set of measure zero.

COROLLARY 2. *The integral of a function $f(x)$ summable on ab,*

$$g(e) = \int_e f(x)dx, \tag{34}$$

has $f(x)$ as its unique derivative almost everywhere.

For the function $g(e)$ is well defined on the normal class $\mathcal{E}$ including all measurable sets on ab. The equation (34) is of the same type as (33) and it follows that all four of the derivative numbers of $g(e)$ are equal to $f(x)$ almost everywhere. This corollary is the justification of the property 5) of Theorems 9 and 15, for the case when e is the interval ax.

COROLLARY 3. *A necessary and sufficient condition that $g(e)$ shall be the integral* (34) *of a function $f(x)$ summable on ab, is that $g(e)$ be additive and absolutely continuous on the normal class on which it is defined.*

The condition is clearly necessary since by Theorem 15 every integral of a function summable on ab is additive and absolutely continuous on the totality $\mathcal{E}$ of measurable sets on ab. On the other hand such a function on a normal class $\mathcal{E}$ is expressible in the form (33), with $g(eE)$ vanishing because g is absolutely continuous and eE has measure zero.

COROLLARY 4. *The function $g(eE)$ is called the function of singularities of g. If E' is a set of measure zero including E then $g(eE) = g(eE')$ for every e. If A is a denumerable set of intervals enclosing E then*

$$g(eE) = \lim_{mA=0} g(eA). \tag{35}$$

If E_1 and E_2 are the parts of E on which all four derivative numbers of g are $+\infty$, $-\infty$, respectively, then

$$g(eE) = g(eE_1) + g(eE_2) \tag{36}$$

and the functions on the right satisfy the inequalities

$$(37) \qquad g(eE_1) \geqq 0, \quad g(eE_2) \leqq 0.$$

Each of the functions in (36) *has a unique derivative equal to zero almost everywhere, and each vanishes if its argument contains no perfect subset.*

To prove the first statement of the corollary one has only to substitute eE' for e in equation (33). A similar substitution of eA for e justifies the formula (35). The inequalities (37) are immediate consequences of Lemma 8. The fact that $g(eE)$ has derivative zero almost everywhere is evident because $g(e)$ and the integral in equation (33) both have derivatives equal to Λ almost everywhere. If the expression (36) is substituted in equation (33) it follows that $g(eE_2)$ is the function of singularities of $g(e) - g(eE_1)$, and hence must also have derivative zero almost everywhere. A similar argument holds for $g(eE_1)$.

The complement of eE can be enclosed in the interior of a denumerable set of intervals A in such a way that

$$(38) \qquad | g(eE) - g(CA) | = | g(A) - g(CeE) | < \epsilon.$$

If the intervals of A are thought of as not including their end points, the set CA is necessarily closed* and the sum of a denumerable set and a perfect set.† But CA is interior to eE and consequently denumerable if eE contains no perfect subset. It follows that $g(CA)$ is zero, since g vanishes on a set consisting of a single point, and CA is the sum of a denumerable number of such sets. The inequality (38) now shows that $g(eE)$ is zero since it must be true for every ϵ. Similar arguments are applicable for $g(eE_1)$ and $g(eE_2)$.

It is a consequence of the Lemma 8, or also of the formulas (33) and (37), that $g(e)$ is necessarily positive or zero on a set e where a derivative number Λ of g is non-negative. If $\Lambda > 0$ on a subset of e of measure greater than zero, then $g(e) > 0$. Consequently on an arbitrary set e the function $g(e)$ attains its greatest value on the subset of e where $\Lambda \geqq 0$, and it follows readily that the total variation of $g(e)$ is given by the formula

$$(39) \qquad T(e) = g(eE_1) - g(eE_2) + \int_e | \Lambda | \, dx.$$

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* VII, vol. 1, 3d edition, p. 54.

† Ibid., p. 52.

§ 9. *Applications of the Preceding Theorems to Functions $G(x)$, and an Example.*

The results described in the preceding section justify a series of notable properties of a function $G(x)$ of limited variation on an interval ab, and the formula (33) in particular interpreted for $G(x)$, gives extensions of the well-known fundamental theorem of the integral calculus which have many important applications.

In the first place every function $G(x)$ of limited variation on ab satisfies the relation

$$G(x) - G(a) = g(\omega) \qquad (\omega = \text{the interval } ax)$$

with a function $g(e)$ as in § 8 (Theorem 18), and has a unique derivative $F(x)$ almost everywhere (Theorem 19, Corollary 1). If arbitrary finite values are assigned to $F(x)$ at the set of points of measure zero where the derivative of $G(x)$ is not defined, then $F(x)$ is summable (Lemma 7). The variation $V(x)$ of $G(x)$ on the portion between a and x of the set E of singularities where $F(x)$ is $+\infty$ or $-\infty$, is defined to be the limit

$$(40) \qquad \lim_{mA=0} \sum_{\alpha} \Delta_{\alpha} G = \lim_{mA=0} g(A),$$

where A is a denumerable set of non-overlapping intervals α enclosing the part of E on ax, and where for an interval α with end points x_1 and x_2

$$\Delta_{\alpha} G = G(x_2) - G(x_1).$$

This limit surely exists (Theorem 19, Corollary 4), and from formula (33)

$$(41) \qquad G(x) - G(a) = V(x) + \int_{ax} F(x)dx.$$

The customary theorem of the integral calculus is that for every function $G(x)$ which has a bounded derivative $F(x)$ integrable on the interval ab in the sense of Riemann

$$(42) \qquad G(x) - G(a) = \int_{ax} F(x)dx.$$

The formula (41) is a generalization of this for functions of limited variation, and has precisely the form (42) when the set of singularities E contains no perfect subset (Theorem 19, Corollary 4), in particular when a derivative number of $G(x)$ is everywhere finite valued. It is provable furthermore that every function $G(x)$ which has a derivative number $F(x)$ finite valued and summable on ab, is necessarily of limited variation on ab, and therefore satisfies formula (42). For let $\Phi(x)$ be a major function for $|F(x)|$. Then each of $\Phi(x)$ and $\Phi(x) - G(x)$ has a derivative number positive or zero, and is necessarily increasing on ab. Consequently

$$G(x) = \Phi(x) - [\Phi(x) - G(x)]$$

is the difference of two monotonically increasing functions.

A function $G(x)$ is said to be absolutely continuous on ab if the limit (40) is zero for all sets A irrespective of their relationship to E. A necessary and sufficient condition that $G(x)$ shall be expressible in the form (42) in terms of a summable function $G(x)$, is that $G(x)$ be absolutely continuous on ab (Theorem 19, Corollary 3). An integral of a summable function of the form in (42) is necessarily of limited variation, as one may see by replacing e by the interval ax in the formula (26), and its total variation on the interval ab is

$$T = \int_{ab} |F|\, dx,$$

according to formula (39). Furthermore such an integral has $F(x)$ as its derivative almost everywhere (Theorem 19, Corollary 2). If $G(x)$ is absolutely continuous and has a derivative number zero almost everywhere, then $G(x)$ must be constant, as one sees from formula (42).

The problem of determining whether or not a given function $F(x)$ is a derivative or a derivative number of a continuous function $G(x)$ was one of those which Lebesgue first considered. If $F(x)$ is continuous the function $G(x)$ defined by equation (42) is always a continuous anti-derivative of $F(x)$. But if $F(x)$ is discontinuous it may not be a derivative number of any continuous function whatsoever. A very simple example suffices to show this. Let $F(0) = 1$ and $F(x) = 0$ for $0 < x \leqq 1$. This function is bounded and measurable, and a function $G(x)$ which has $F(x)$ as a derivative number must therefore be expressible in the form (42). The function $G(x)$ so defined is clearly a constant and has everywhere the unique derivative zero. The method of procedure suggested by this example is applicable in general to finite valued summable functions. If such a function $F(x)$ is a derivative number of $G(x)$, then the latter must be expressible in the form (42). It has $F(x)$ as derivative almost everywhere, and it may be possible by actual test to determine whether or not this relationship is universally valid.

It may be of interest to consider an example of a function $G(x)$ of limited variation on the interval 01, for which the variation V in formula (41) is not zero.* Let $G(0) = 0$, $G(1) = 1$, and $G(x) = 1/2$ on the interval $1/3 < x < 2/3$. On the middle thirds of the two remaining intervals $G(x)$ is to have the values 1/4, 3/4 respectively. The process may be continued indefinitely. After division of 01 into 3^n equal parts the sum of the intervals on which $G(x)$ is not defined is $(2/3)^n$, so that the total set Ξ where G is not defined by this process has measure zero. Every point ξ of Ξ is therefore a limit point of points x of the set X where G has already been specified, and it is not difficult to show that $G(x)$ has a definite limit as x approaches ξ on the set X. This limit is to be taken as the value $G(\xi)$. The function $G(x)$ so determined on the whole interval 01 is monotonically increasing, and has derivative zero everywhere except at the points of the set Ξ of measure zero. Hence in formula (41) the integral vanishes and

$$1 = G(1) - G(0) = V(1).$$

It should be emphasized here in closing that the properties of functions of limited variation described in the paragraphs just above are a small part only of the applications of Lebesgue integrals. Every discussion involving definite integrals may give rise to important new results if the integrals are reinterpreted in the sense of Lebesgue, If an arc in space is rectifiable the functions $x(t)$, $y(t)$, $z(t)$ $(t_1 \leqq t \leqq t_2)$ defining it are of limited variation and have derivatives almost everywhere. The length of the arc is the value of the integral

$$\int_e \sqrt{x'^2 + y'^2 + z'^2}\, dt$$

taken in the Lebesgue sense over the set e on which the three derivatives exist simultaneously.† Vallée Poussin has enumer-

* I am indebted to Mr. K. W. Lamson for the suggestion that a function of this type would illustrate the formula. See also III, p. 13.

† III, p. 125.

ated the various hypotheses under which the formula

$$\int_e f dx = \lim_{n=\infty} \int_e s_n dx$$

holds when $\lim s_n = f$.* The conditions described in the Corollary to Theorem 9 can be greatly generalized to cover cases in which the functions s_n are not necessarily bounded in in their totality. The formulas for integration by parts, and for transformation of simple and double integrals, have new and more powerful interpretations in the Lebesgue theory,† and a multiple integral is reduced to an iterated integral by a particularly beautiful theorem.‡ If a function $f(x, y)$ is measurable and bounded on a two-dimensional set E then

$$\int \int_E f(x, y) dx dy = \int_X \int_{Y_x} f(x, y) dy dx$$

where X is the projection of E on the x-axis, and Y_x is the section of E on the ordinate over the abscissa x. The inner integral on the right does not exist for every x in the projection X, but it turns out that this does not affect its integrability over X since the set of points where it is not well-defined has measure zero. The problem of determining a function whose Fourier constants are given is really the problem of the summation of a Fourier series which may or may not be convergent according to the usual definitions. Riesz and Fischer§ with the help of the Lebesgue theory proved the existence of a solution under very general circumstances which were later still further generalized by Riesz.‖ The theory of Fourier series has been retouched in many other places also as a result of contact with the new integrals.¶ But it is impossible to list here in detail the rapidly increasing number of applications. More important than any one of them by itself are the new habits of thinking of and dealing with discontinuities too serious to be handled by the older forms of integration, and these will be the permanent legacies of the theory of Lebesgue.

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* VI, p. 443; V, vol. 1, 3d edition, p. 263; VII, p. 48.

† VI, pp. 465, 500; V, vol. 1, 3d edition, pp. 279 ff.

‡ VII, p. 50; V, vol. 2, 2d edition, p. 120.

§ See, for example, Lalesco, Introduction à la théorie des équations intégrales, p. 95.

‖ Loc. cit.

¶ See Lebesgue, Leçons sur les séries trigonométriques; and V, vol. 2, 2d edition, pp. 130 ff.

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NOTES.

THE July number (volume 18, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Set of independent postulates for betweenness," by E. V. HUNTINGTON and J. R. KLINE; "Haskins's momental theorem and its connection with Stieltjes's problem of moments," by E. B. VAN VLECK; "Point sets and allied Cremona groups (part III)," by A. B. COBLE; "On the second derivatives of the extremal integral for the integral $\int F(y;\, y')dt$," by ARNOLD DRESDEN; "Concerning singular transformations B_k of surfaces applicable to quadrics," by LUIGI BIANCHI; "Types of (2, 2) point correspondences between two planes," by F. R. SHARPE and VIRGIL SNYDER.

THE July number (volume 39, number 3) of the *American Journal of Mathematics* contains: "A concomitant curve of the plane quartic," by TERESA COHEN; "On two related transformations of space curves," by W. C. GRAUSTEIN; "The space problem of the calculus of variations in terms of angle," by P. R. RIDER; "Derivation of the complementary theorem from the Riemann-Roch theorem," by S. BEATTY; "On the equivalence of relations $K_{q_1 q_2 m}$," by E. W. CHITTENDEN; "Some properties of certain finite algebras," by EDWARD KIRCHER; "The primitive groups of class 15," by W. A. MANNING;